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Embeddability into relational lattices is undecidable

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ABSTRACT

The natural join and the inner union operations combine relations of a database. Tropashko and Spight realized that these two operations are the meet and join operations in a class of lattices, known by now as the relational lattices. They proposed then lattice theory as an algebraic approach to the theory of databases alternative to the relational algebra. Litak et al. proposed an axiomatization of relational lattices over the signature that extends the pure lattice signature with a constant and argued that the quasiequational theory of relational lattices over this extended signature is undecidable.

We prove in this paper that embeddability is undecidable for relational lattices. More precisely, it is undecidable whether a finite subdirectly-irreducible lattice can be embedded into a relational lattice. Our proof is a reduction from the coverability problem of a multimodal frame by a universal product frame and, indirectly, from the representability problem for relation algebras.

As corollaries we obtain the following results: the quasiequational theory of relational lattices over the pure lattice signature is undecidable and has no finite base; there is a quasiequation over the pure lattice signature which holds in all the finite relational lattices but fails in an infinite relational lattice.

1. Introduction

The natural join and the inner union operations combine relations (i.e. tables) of a database. Most of today's web programs query their databases making repeated use of the natural join and of the union, of which the inner union is a mathematically well behaved variant. Tropashko and Spight realized [1,2] that these two operations are the meet and join operations in a class of lattices, known by now as the class of relational lattices. They proposed then lattice theory as an algebraic approach, alternative to Codd's relational algebra [3], to the theory of databases.

An important first attempt to axiomatize these lattices is due to Litak, Mikulás, and Hidders [4]. These authors propose an axiomatization, comprising equations and quasiequations, over a signature that extends the pure lattice signature with a constant, the header constant. A main result of that paper is that the quasiequational theory of relational lattices is undecidable over this extended signature. Their proof mimics Maddux's proof that the equational theory of cylindric algebras of dimension $n \geq 3$ is undecidable [5].

We have investigated in [6] equational axiomatizations for relational lattices using as tool the duality theory for finite lattices developed in [7]. A conceptual contribution from [6] is to make explicit the similarity between the developing theory of relational lattices and the well established theory of combination of modal logics, see e.g. [8]. This was achieved on the syntactic side, but also on the semantic side, by identifying some key properties of the structures dual to the finite atomistic

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lattices in the variety generated by the relational lattices, see [6, Theorem 7]. These properties make the dual structures into frames for commutator multimodal logics in a natural way.

In this paper we exploit this similarity to transfer results from the theory of multidimensional modal logics to lattice theory. Our main result is that *it is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice*. We prove this statement by reducing to it the coverability problem of a frame by a universal $S5^3$-product frame, a problem shown to be undecidable in [9]. As stated there, the coverability problem is—in light of standard duality theory—a direct reformulation of the representability problem of finite simple relation algebras, problem shown to be undecidable by Hirsch and Hodkinson [10].

Our main result and its proof allow us to derive further consequences. Firstly, we refine the undecidability theorem of [4] and prove that *the quasiequational theory of relational lattices over the pure lattice signature is undecidable* as well and *has no finite base*. Then we argue that *there is a quasiequation that holds in all the finite relational lattices, but fails in an infinite one*. For the latter result, we rely on the work by Hirsch, Hodkinson, and Kurucz [9] who constructed a finite 3-multimodal frame which has no finite p-morphism from a finite universal $S5^3$-product frame, but has a p-morphism from an infinite one. On the methodological side, we wish to point out our use of generalized ultrametric spaces to tackle these problems. A key idea in the proof of the main result is the characterization of universal $S5^A$-product frames as pairwise complete generalized ultrametric spaces with distance valued in the Boolean algebra $P(A)$, a characterization that holds when A is finite.

The paper is structured as follows. We recall in Section 2 some definitions and facts on frames and lattices. Relational lattices are introduced in Section 3. In Section 4 we give an outline of the proof of our main technical result—the undecidability of embeddability of a finite subdirectly-irreducible lattice into a relational lattice—and derive from it the other results. In Section 5 we show how to construct a lattice from a frame and use functoriality of this construction to argue that such lattice embeds into a relational lattice whenever the frame is a p-morphic image of a universal product frame. The proof of the converse statement is carried out in Section 8. The technical tools needed to prove the converse are developed Sections 6 and 7. The theory of generalized ultrametric spaces over a powerset Boolean algebra and the aforementioned characterization of $S5^A$-product frames as pairwise complete spaces over $P(A)$ appear in Section 6. In Section 7 we study embeddings of finite subdirectly-irreducible lattices into relational lattices and prove that we can assume that these embeddings preserve bounds. This task is needed so to exclude the constants $\bot$ and $\top$ (denoting the bounds) from the signature of lattice theory.

2. Frames and lattices

Frames Let A be a set of actions. An *A-multimodal frame* (briefly, an *A-frame* or a *frame*) is a structure $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\}\rangle$ where, for each $a \in A$, R_a is a binary relation on $X_{\mathfrak{F}}$. We say that an A-frame is S4 if each R_a is reflexive and transitive. If $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are two A-frames, then a *p-morphism* from $\mathfrak{F}_0$ to $\mathfrak{F}_1$ is a function $\psi : X_{\mathfrak{F}_0} \longrightarrow X_{\mathfrak{F}_1}$ such that, for each $a \in A$,

- if xR_ay, then $\psi(x)R_a\psi(y)$,
- if $\psi(x)R_az$, then xR_ay for some y with $\psi(y) = z$.

Let us mention that A-multimodal frames and p-morphisms form a category.

A frame $\mathfrak{F}$ is said to be *rooted* (or *initial*, see [11]) if there is $f_0 \in X_{\mathfrak{F}}$ such that every other $f \in X_{\mathfrak{F}}$ is reachable from f_0. We say that an A-frame $\mathfrak{F}$ is *discriminating* if, for each $a \in A$, there exists $f, g \in X_{\mathfrak{F}}$ such that $f \neq g$ and fR_ag. If $G = (V, D)$ is a directed graph, then we shall say that G is rooted if it is rooted as a unimodal frame.

A particular class of frames we shall deal with are the *universal $S5^A$-product frames*. These are the frames $\mathfrak{U}$ with $X_{\mathfrak{U}} = \prod_{a\in A} X_a$ and xR_ay if and only if $x_i = y_i$ for each $i \neq a$, where $x := \langle x_i \mid i \in A\rangle$ and $y := \langle y_i \mid i \in A\rangle$.

Let $\alpha \subseteq A$, $\mathfrak{F}$ be an A-frame, $x, y \in X_{\mathfrak{F}}$. An *$\alpha$-path* from x to y is a sequence $x = x_0R_{a_0}x_1 \dots x_{k-1}R_{a_{k-1}}x_k = y$ with $\{a_0, \dots, a_{k-1}\} \subseteq \alpha$. We use then the notation $x \xrightarrow{\alpha} y$ to mean that there is an α-path from x to y. Notice that if $\mathfrak{F}$ is an S4 A-frame, then $x \xrightarrow{\{a\}} y$ if and only if xR_ay.

Orders and lattices We assume some basic knowledge of order and lattice theory as presented in standard monographs [12,13]. Most of the tools we use in this paper originate from the monograph [14] and have been further developed in [7].

A *lattice* is a poset L such that every finite non-empty subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$ and a greatest lower bound $\bigwedge X$. A lattice can also be understood as a structure $\mathfrak{A}$ for the functional signature $(\vee, \wedge)$, such that the interpretations of these two binary function symbols both give $\mathfrak{A}$ the structure of an idempotent commutative semigroup, the two semigroup structures being connected by the absorption laws $x \wedge (y \vee x) = x$ and $x \vee (y \wedge x) = x$. Once a lattice is presented as such structure, the order is recovered by stating that $x \leq y$ holds if and only if $x \wedge y = x$.

A lattice L is *complete* if any subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$. It can be shown that this condition implies that any subset $X \subseteq L$ admits a greatest lower bound $\bigwedge X$. A lattice is *bounded* if it has a least element $\bot$ and a greatest element $\top$. A complete lattice (in particular, a finite lattice) is bounded, since $\bigvee \emptyset$ and $\bigwedge \emptyset$ are, respectively, the least and greatest elements of the lattice.

If P and Q are partially ordered sets, then a function $f : P \longrightarrow Q$ is *order-preserving* (or *monotone*) if $p \leq p'$ implies $f(p) \leq f(p')$. If L and M are lattices, then a function $f : L \longrightarrow M$ is a *lattice morphism* if it preserves the lattice operations