Doctopic: Algorithms, automata, complexity and games ARTICLE IN PRESS TCS:11588

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

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Edge coloring of planar graphs without adjacent 7-cycles ☆

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ARTICLE INFO

Article history:
Received 20 August 2017
Received in revised form 27 April 2018
Accepted 1 May 2018
Available online xxxx
Communicated by L.M. Kirousis

Keywords:
Edge coloring
Planar graph
Cycle
Class 1

ABSTRACT

A graph is said to be of class 1 if its edge chromatic number is equal to the maximum degree of this graph. Let G be a planar graph with maximum degree $\Delta \geq 6$ and without adjacent 7-cycles, then G is of class 1.

1. Introduction

All graphs considered here are finite, simple and undirected. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$. For vertices $u, v, w \in V(G)$, let $E_G(u)$ or $E(u)$ be the set of edges incident with u, $N(u)$ the set of vertices adjacent to u, and $N(u,v) = N(u) \bigcup N(v)$, $N(N(u)) = \{w | vw \in E, v \in N(u)\}$, $N(N(u,v)) = N(N(u)) \bigcup N(N(v))$. The degree of v in G, denoted by $d_G(v)$ or $d(v)$, is the cardinality of $E(v)$. A *k-vertex*, *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. A k (or k^+)-vertex adjacent to a vertex x is called a k (or k^+)-neighbor of x. A k-cycle is a cycle of length k. Two cycles sharing at least a common edge are said to be adjacent.

A graph is *k-edge-colorable*, if its edges can be colored with k colors such that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing showed that if G is a graph with maximum degree Δ, then $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. A graph G is said to be of *class* 1 if $\chi'(G) = \Delta$, and of *class* 2 if $\chi'(G) = \Delta + 1$. A graph G is *critical* if it is connected and of class 2, and $\chi'(G-e) < \chi'(G)$ for any edge e of G. A critical graph with maximum degree Δ is called a *Δ-critical graph*. It is clear that every critical graph is 2-connected.

A *planar graph* is a graph which can be embedded in the plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. If a connected graph G is embedded in the plane in this way, it is called a *plane graph*. For planar graphs, more is known. As noted by Vizing [1], if C_4, K_4, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for $\Delta \in \{2, 3, 4, 5\}$. He proved that every planar graph

☆ This work was partially supported by NSFC (No. 11271006, No. 11401386), Shandong Provincial Natural Science Foundation, China (No. ZR2017BA009), General plan of science and technology program of Shandong higher education institutions (No. J17KA168) and Talent introduction research project of Shangdong Woman's University (No. 2016YJRC12).

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https://doi.org/10.1016/j.tcs.2018.05.006
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Please cite this article in press as: W. Zhang, J.-L. Wu, Edge coloring of planar graphs without adjacent 7-cycles, Theoret. Comput. Sci. (2018), https://doi.org/10.1016/j.tcs.2018.05.006

with $\Delta \geq 8$ is of class 1 (There are more general results, see [2] and [4]) and then conjectured that every planar graph with maximum degree 6 or 7 is of class 1. The case $\Delta = 7$ for the conjecture has been verified by Zhang [7] and, independently, by Sanders and Zhao [6]. The case $\Delta = 6$ remains open, but some partial results are obtained. Theorem 16.3 [1] stated that a planar graph with the maximum degree Δ and the girth g is of class 1 if $\Delta \geq 3$ and $g \geq 8$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 5$ and $g \geq 4$. Lam, Liu, Shiu and Wu [3] proved that a planar graph G is of class 1 if $\Delta \geq 6$ and no two 3-cycles of G sharing a common vertex. Zhou [8] obtained that every planar graph with $\Delta \geq 6$ and without 4 or 5-cycles is of class 1. Ni [5] proved that a planar graph G is of class 1 if $\Delta \geq 6$ and any two k-cycles of length at most 6 are not adjacent.

In this paper, we get the following result.

Theorem 1. *Suppose G is a planar graph without adjacent 7-cycles. If $\Delta \geq 6$, then G is of class 1.*

2. Proof of Theorem 1

To prove our result, we will introduce some known lemmas.

Lemma 2. [6,7] *If G is a planar graph with $\Delta(G) \geq 7$, then G is of class 1.*

Lemma 3. (Vizing's Adjacency Lemma [1]) *Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with $d(v) = k$.*

(*a*) *If $k < \Delta$, then u is adjacent to at least $\Delta - k + 1$ vertices of degree Δ;*
(*b*) *If $k = \Delta$, then u is adjacent to at least two vertices of degree Δ.*

Lemma 4. [7] *Let G be a Δ-critical graph, $uv \in E(G)$ and $d(u) + d(v) = \Delta + 2$. Then*

(*a*) *every vertex of $N(\{u, v\}) \setminus \{u, v\}$ is a Δ-vertex;*
(*b*) *every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is of degree at least $\Delta - 1$;*
(*c*) *if $d(u), d(v) < \Delta$, then every vertex of $N(N(\{u, v\})) \setminus \{u, v\}$ is a Δ-vertex.*

Lemma 5. [6] *No Δ-critical graph has distinct vertices x, y, z such that x is adjacent to y and z, $d(z) < 2\Delta - d(x) - d(y) + 2$, and xz is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y.*

To be convenient, for a plane graph G, let $F(G)$ be the face set of G. A face of a graph is said to be *incident* with all edges and vertices in its boundary. The degree of a face f, denoted by $d_G(f)$ is the number of edges incident with f where each cut edge is counted twice. A k-, k^+-face is a face of degree k, at least k. A k-face of G is called an $(i_1, i_2, \cdots, i_k)$-face if the vertices in its boundary are of degrees $i_1, i_2, \cdots, i_k$ respectively. A 3-face is denoted by $[x, y, z]$ if it is incident with distinct vertices x, y, z and $d(x) \leq d(y) \leq d(z)$. A 4-face $f = [w, v, x, y]$ is called *special* if $d(x) = 2$ and v, x, y form a 3-face. For a vertex $v \in V(G)$, we denote by $d_k(v)$, $d_{k^+}(v)$ the number of k-neighbors, k^+-neighbors of v and $f_k(v)$ the number of k-faces incident with v.

Lemma 6. *Let G be a Δ-critical graph. For every 6-vertex $v \in V(G)$, if $d_2(v) = 1$ (say is v_1) and if any two 7-cycles are not adjacent in G, then*

(*a*) *If $f_3(v) = 3$, then $f_{5^+}(v) \geq 1$;*
(*b*) *If $f_3(v) = 4$ and v_1 are incident with one 3-face and one 5-face, then $f_{8^+}(v) = 1$;*
(*c*) *If $f_3(v) = 4$ and v_1 are incident with one 3-face and one 6-face, then $f_{7^+}(v) = 1$.*

Proof. Let $v_1, v_2, \cdots, v_6$ be neighbors of v of G in an anticlockwise order. Let f_i of G be face incident with v, v_i and v_{i+1}, for all i such that $i \in \{1, 2, \cdots, 6\}$. Note that all the subscripts in the paper are taken modulo 6.

(a) There are some cases by symmetry. (1) f_1, f_2 and f_3 are 3-faces. (2) f_1, f_2 and f_4 are 3-faces. (3) f_1, f_2 and f_5 are 3-faces. (4) f_1, f_3 and f_4 are 3-faces. (5) f_1, f_3 and f_5 are 3-faces. (6) f_1, f_4 and f_5 are 3-faces. (7) f_2, f_3 and f_4 are 3-faces. (8) f_2, f_3 and f_5 are 3-faces. Now we prove that (1). If $d(f_4) \geq 5$, then $f_{5^+}(v) \geq 1$; otherwise $d(f_4) = 4$. If $d(f_6) \geq 5$, then $f_{5^+}(v) \geq 1$; otherwise $d(f_6) = 4$. If $d(f_5) = 4$, then there are adjacent 7-cycles. So $d(f_5) \geq 5$ and $f_{5^+}(v) \geq 1$. (2)–(8) are similar to be proved as (1), we omit here.

(b) Since v_1 are incident with one 3-face and one 5-face, so $d(f_1) = 3$ and $d(f_6) = 5$. There are some cases by symmetry. (1) f_2, f_3 and f_4 are 3-faces. (2) f_2, f_3 and f_5 are 3-faces. (3) f_2, f_4 and f_5 are 3-faces. (4) f_3, f_4 and f_5 are 3-faces. Now we prove that (1). If $d(f_5) \in \{4, 5, 6, 7\}$, then there are adjacent 7-cycles. So $d(f_5) \geq 8$ and $f_{8^+}(v) = 1$. (2)–(4) are similar to be proved as (1), we omit here.

(c) It is similar to be proved as (b), we omit here. □

Lemma 7. *Let G be a Δ-critical graph such that any two 7-cycles are not adjacent in G. For every 6-vertex $v \in V(G)$, if $d_2(v) = 1$, $f_3(v) = 4$ and v is incident with a special 4-face $f = [u, v, w, z]$ such that $d(u) = 2$, then $f_3(z) \leq 3$ and $f_{5^+}(z) \geq 2$.*

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