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Context-free commutative grammars with integer counters and resets

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A B S T R A C T

We study the computational complexity of reachability, coverability and inclusion for extensions of context-free commutative grammars with integer counters and reset operations on them. Those grammars can alternatively be viewed as an extension of communication-free Petri nets. Our main results are that reachability and coverability are inter-reducible and both NP-complete. In particular, this class of commutative grammars enjoys semi-linear reachability sets. We also show that the inclusion problem is, in general, coNEXP-complete and already Π_2^{P}-complete for grammars with only one non-terminal symbol. Showing the lower bound for the latter result requires us to develop a novel Π_2^{P}-complete variant of the classic subset sum problem.

1. Introduction

This paper studies the computational complexity of certain decision problems for extensions of context-free commutative grammars with integer counters and reset operations on them. The motivation for our work comes from the close relationship of such grammars with subclasses of Petri nets. For presentational purposes, we begin with introducing the decision problems we consider in terms of Petri nets.

Petri nets, or equivalently Vector Addition Systems with States (VASS), are a prominent and appealing class of infinite-state systems, from both theoretical and practical perspectives. On the one hand, their high level of abstraction allows them to be used as a mathematical model with well-defined semantics in a wide range of application domains, in particular but not limited to the verification of concurrent programs, see, e.g., [1]. On the other hand, for half a century Petri nets have provided a pool of challenging and intricate decision problems and questions about their structural properties. One of the most important and well-known instances is the question about the computational complexity of the reachability problem for Petri nets, which has attracted the attention of generations of researchers without, however, having been fully resolved.

A Petri net comprises a finite set of places with a finite number of transitions. Places may contain a finite number of tokens, and a transition can consume tokens from places, provided sufficiently many are present, and then add a finite number of tokens to some places. In the VASS setting, places are referred to as counters and we will often use these terms interchangeably in this paper. A configuration of a Petri net is a marking of its places, which is just a function $\boldsymbol{m}: \text{Places} \to \mathbb{N}$

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or, equivalently, a vector of natural numbers whose components are indexed by elements from Places. The most prominent decision problems for Petri nets are reachability, coverability and inclusion. Given configurations $\boldsymbol{m}$ and $\boldsymbol{n}$ of a Petri net $\mathcal{A}$, reachability is to decide whether there is a sequence of transitions of $\mathcal{A}$ whose effect transforms $\boldsymbol{m}$ into $\boldsymbol{n}$. Coverability asks whether there is a transition sequence from $\boldsymbol{m}$ to a configuration that is "above" $\boldsymbol{n}$, i.e., a path to some configuration $\boldsymbol{n}'$ such that $\boldsymbol{n}' \geq \boldsymbol{n}$, where $\geq$ is interpreted component-wise. Finally, given Petri nets $\mathcal{A}$ and $\mathcal{B}$ with the same set of places, inclusion asks whether the set of markings reachable in $\mathcal{A}$ is contained in the set of those reachable in $\mathcal{B}$. All of these problems have been extensively studied in the literature. One of the earliest results was obtained by Lipton, who showed that reachability and coverability are EXPSPACE-hard [2]. Subsequently, Rackoff established a matching upper bound for coverability [3], and Mayr showed that reachability is decidable [4]. This result was later refined [5,6] and shown in a different way in [7], and an actual complexity-theoretic upper bound, namely membership in $\mathbf{F}_{\omega^3}$, a level of the fast-growing hierarchy, was only recently established [8]. For inclusion, it is known that this problem is in general undecidable [9] and Ackermann ($\mathbf{F}_\omega$)-complete when restricting to Petri nets with a finite reachability set [10].

For some application domains, standard Petri nets are not sufficiently expressive. For instance, as discussed, e.g., in [11], the verification of concurrent finite-state shared-memory programs requires additional operations on places such as transfers, where the content of one place can be copied to another one. Another example is the validation of business processes, which requires reset operations on places, i.e., a special kind of transitions which assign the value zero to some place [12]. The computational price for these extensions is high: reachability in the presence of any such extension becomes undecidable [13,14], while the complexity of coverability increases significantly to Ackermann ($\mathbf{F}_\omega$)-completeness in the presence of resets [15].

One of the main sources of the high complexity of decision problems for Petri nets and their extensions is the restriction that the places contain a *non-negative* number of tokens. This restriction enables one to enforce an order in which transitions can be taken, which is at the heart of many hardness proofs. In this paper, we relax this restriction and study the computational complexity of decision problems for a subclass of Petri nets, where the nets have additional counters that range over the integers and can be reset and where transitions are also structurally restricted. One advantage of this class is the decidability and a much lower computational complexity of standard decision problems when compared to usual Petri nets with reset operations.

Our contribution.

The main focus of this paper is the computational complexity of reachability, coverability and inclusion for so-called communication-free Petri nets extended with integer counters and resets, and for subclasses thereof. A communication-free Petri net is a Petri net in which every transition can remove a token from at most one place. An important property of communication-free Petri nets is that their sets of reachable markings are semi-linear [16–18], meaning in particular that they are closed under all Boolean operations (this is not the case for general Petri nets [19]). Communication-free Petri nets are essentially equivalent[3] to context-free commutative grammars, or basic parallel processes, and have extensively been studied in the literature [17,18,20–25]. For technical convenience we adopt the view of communication-free Petri nets as context-free commutative grammars in the technical part of this paper.

As our first main result, we show that context-free commutative grammars can be extended by a finite number of integer counters, i.e., counters that range over the integers and can be reset by transitions, while retaining NP-completeness of reachability and coverability, as well as preserving semi-linearity of the reachability set. This is achieved by showing that the reachability set of our extended class can be defined by a formula in existential Presburger arithmetic of polynomial size. The characterization obtained in this way can then be used in order to show coNEXP-completeness of the inclusion problem by application of complexity bounds for Presburger arithmetic.

Our second main result is a more refined analysis of the complexity of the inclusion problem. We show that even in the structurally simplest case of context-free commutative grammars with integer counters and *without any* control structure, i.e., a singleton non-terminal alphabet, the inclusion problem is hard for the second level of the polynomial hierarchy and, in fact, Π_2^{P}-complete. In essence, this problem is equivalent to asking, given two integer matrices A, B and a vector $\boldsymbol{v} \in \mathbb{N}^d$, whether for all $\boldsymbol{x} \in \mathbb{N}^m$ there exists some $\boldsymbol{y} \in \mathbb{N}^n$ such that $A \cdot \boldsymbol{x} + B \cdot \boldsymbol{y} = \boldsymbol{v}$. We prove hardness of this problem by developing a new Π_2^{P}-complete variant of the classical Subset Sum problem, which we believe is a contribution of independent interest.

This paper is an extended version of our conference paper [26], which appeared in the proceedings of the 8th International Workshop on Reachability Problems (RP 2014) held in Oxford, UK, in September 2014. It extends the results from [26] by considering a more general model: context-free commutative grammars with integer counters and resets instead of integer vector addition systems with states and resets considered in [26]; we also provide full proofs. Moreover, in [26] we left as an open question the precise complexity of the aforementioned Π_2^{P}-complete inclusion problem, which we could only show to be NP-hard and in Π_2^{P}. This question is now resolved in this paper.

Related work.

Apart from the related work mentioned above, closely connected to the problems considered in this paper is the work by Kopczyński and To [23] and Kopczyński [24]. In their work, the complexity of various decision problems for context-free commutative grammars and subclasses thereof has been studied when the number of alphabet symbols (which roughly

[3] This will be made more precise in Section 2.4.