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Compressed cliques graphs, clique coverings and positive zero forcing

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A R T I C L E I N F O

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A B S T R A C T

Zero forcing parameters, associated with graphs, have been studied for over a decade, and have gained popularity as the number of related applications grows. In particular, it is well-known that such parameters are related to certain vertex coverings. Continuing along these lines, we investigate positive zero forcing within the context of certain clique coverings. A key object considered here is the *compressed cliques graph*. We study a number of properties associated with the compressed cliques graph, including: uniqueness, forbidden subgraphs, connections to Johnson graphs, and positive zero forcing.

1. Introduction

Suppose that G is a simple finite graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use $\{u, v\}$ to denote an edge with endpoints u and v. Further, for a graph $G = (V, E)$ and $v \in V$, the vertex set $\{u : \{u, v\} \in E\}$ is the *neighbourhood* of v, denoted as $N_G(v)$, and the size of the neighbourhood of v is called the *degree* of v. For $V' \subseteq V$, the vertex set $\{x : \{x, y\} \in E, x \in V \setminus V'$ and $y \in V'\}$ is the *neighbourhood* of V', denoted as $N_G(V')$. Also, we let the set $N_G[v] = \{v\} \cup N_G(v)$ denote the *closed neighbourhood* of the vertex v. We use $G[V']$ to denote the subgraph induced by V', which consists of all vertices of V' and all of the edges in G that contain only vertices from V'. We use $G - v$ to denote the subgraph induced by $V \setminus \{v\}$. Unless otherwise stated all graphs considered here are connected.

For an integer $n \geq 1$, we let K_n denote the complete graph on n vertices. We will also refer to a complete graph on n vertices as a *clique* on n vertices. A *cycle* on n vertices $\{v_1, v_2, \ldots, v_n\}$ is a graph with edges $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. Such a cycle will also be referred to as an n-cycle or a cycle of length n.

Our interest in this work is to consider how positive zero forcing sets are related to cliques in a graph and, further, clique intersection, and clique coverings. Zero forcing on a graph was originally designed to be used as a tool to bound the maximum nullity associated with collections of symmetric matrices derived from a graph G [1]. Independently, this parameter was also considered in other contexts as the graph infection number in [4] and as the fast-mixed search number

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in [15]. Positive zero forcing was an adaptation of conventional zero forcing to play a similar role for positive semidefinite matrices (see [3]).

Zero forcing in general is a graph colouring problem in which an initial set of vertices are coloured black, while the remaining vertices are coloured white. Using a designated colour rule, the objective is to change the colour of as many white vertices to black as possible. There are two common rules, which are known as zero forcing and positive zero forcing. The process of a black vertex u changing the colour of a white vertex v to black is usually referred to as "u forces v". The size of the smallest initial set of black vertices that will "force" all vertices black is called either the *zero forcing number* or the *positive zero forcing number* of G depending on which rule is used. This number is denoted by either $Z(G)$ or $Z_+(G)$ (again, depending on which rule is used).

Here we are more interested in the behaviour of the positive zero forcing number in connection with cliques and clique coverings in a graph. In particular, we consider $Z_+(G)$, when maximal cliques of G satisfy certain intersection properties. Consequently, we now carefully review some basic terminology associated with positive zero forcing in a graph.

The positive zero forcing rule is also based on a colour change rule similar to the zero forcing colour change rule (see [3] and also [7] and [8]). In this case, suppose G is a graph and B a subset of vertices; we initially colour all of the vertices in B black, while all remaining vertices are designated white. Let $W_1, \dots, W_k$ be the sets of vertices in each of the connected components of G after removing the vertices in B. If u is a vertex in B and w is the only white neighbour of u in the graph induced by the subset of vertices $W_i \cup B$, then u can force the colour of w to black. This rule is called the *positive colour change rule*. The size of the smallest positive zero forcing set of a graph G is denoted by $Z_+(G)$. For all graphs G, since a zero forcing set is also a positive zero forcing set we have that $Z_+(G) \le Z(G)$. A number of facts have been demonstrated for the positive zero forcing number, see, for example, [3]. If a subset S of $V(G)$ is a positive zero forcing set with $|S| = Z_+(G)$, then we refer to S as an *optimal* positive zero forcing set for G.

It is known that by following the sequence of forces throughout the conventional zero forcing process, a path covering of the vertices is derived (see [3, Proposition 2.10] for more details). When the positive colour change rule is applied a vertex can force multiple vertices at the same time. This implies that the positive colour change rule produces a partitioning of the vertices into sets of vertex disjoint induced rooted trees, which we will refer to as *forcing trees*, in the graph.

Given K_n, the complete graph on $n \ge 2$ vertices, it is not difficult to observe that

$$Z_+(K_n) = Z(K_n) = n - 1.$$

Furthermore, even though the parameters Z and Z_+ are not generally monotone on induced subgraphs, it is true that if G contains a clique on k vertices, then both $Z(G)$ and $Z_+(G)$ are at least $k - 1$. So in some sense, cliques in a graph play an important role in determining both zero forcing and positive zero forcing sets in a graph. We explore this correspondence further in this paper. As an example, consider chordal graphs (that is, graph with no induced cycles of length 4 or more). For chordal graphs it is known that the positive zero forcing number is equal to the number of vertices minus the fewest number of cliques that contain all of the edges (see, for example, [9]).

Our paper is divided into eight sections. The next three sections deal with certain types of clique coverings, and properties of an associated graph, called a *compressed cliques graph* that results from these identified clique coverings. Section 5 connects compressed cliques graphs with certain well-studied Johnson graphs, and Section 6 is concerned with forbidden subgraphs associated with compressed cliques graphs. Following this, the next section discusses examples, including a new family of graphs called the *vertex-clique* graph and a related concept we call the *reduced graph*. We conclude with a brief outline for potential future research along these lines.

2. Simply intersecting clique coverings

Recall that a *clique* in a graph is a subset of vertices which induces a complete subgraph. A clique in a graph is *maximal* if no vertex in the graph can be added to it to produce a larger clique. A *clique covering* of a graph is a set of cliques with the property that every edge is contained in at least one of the subgraphs induced by one of the cliques in the set. (Note that unless the graph has isolated points, a clique covering that contains every edge also contains every vertex.) The *size* of a clique covering is the number of cliques in the covering. For a graph G, we denote the size of a smallest clique covering by $\mathrm{cc}(G)$. Further, we call a given clique covering *minimal* if the number of cliques in this covering is equal to $\mathrm{cc}(G)$. Observe that a graph G is a complete graph if and only if $\mathrm{cc}(G) = 1$. Clique coverings and related partitions have been well studied with respect to many properties associated with graphs, see, for example, [13] and see also [2] in the context of maximum nullity and zero forcing. Certainly clique (edge-clique or line-clique) covering have appeared in the literature, but to the best of our knowledge max–min clique coverings have not been studied previously.

A clique covering for a graph G is called a *min–max clique covering* if its size is $\mathrm{cc}(G)$ and every clique in it is maximal. Let G be a graph and let $\{C_1, C_2, \dots, C_\ell\}$ be a min–max clique covering. Any minimal clique covering can be transformed into a min–max clique covering by appropriately adding vertices to the non-maximal cliques.

Proposition 2.1. *For any graph G, there exists a clique covering with size $\mathrm{cc}(G)$ in which every clique is maximal, that is, there is always a min–max clique covering of G.*

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