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Effectivity questions for Kleene's recursion theorem

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ABSTRACT

The present paper investigates the quality of numberings measured in three different ways: (a) the complexity of finding witnesses of Kleene's Recursion Theorem in the numbering; (b) for which learning notions from inductive inference the numbering is an optimal hypothesis space; (c) the complexity needed to translate the indices of other numberings to those of the given one. In all three cases, one assumes that the corresponding witnesses or correct hypotheses are found in the limit and one measures the complexity with respect to the best criterion of convergence which can be achieved. The convergence criteria considered are those of finite, explanatory, vacillatory and behaviourally correct convergence. The main finding is that the complexity of finding witnesses for Kleene's Recursion Theorem and the optimality for learning are independent of each other. Furthermore, if the numbering is optimal for explanatory learning and also allows to solve Kleene's Recursion Theorem with respect to explanatory convergence, then it also allows to translate indices of other numberings with respect to explanatory convergence.

1. Introduction

In constructive mathematics, one not only asks whether a certain object exists but also how complex it is to find or construct this object. Hayashi [15] studied this question of finding or constructing objects as a limiting process where the solution (or programs to compute the solution) are approximated in the limit in the same way as inductive inference approximates the program of a concept to be learnt. In other words, Hayashi [15] carried over the convergence criteria from inductive inference and applied them to finding solutions to specific mathematical tasks which cannot be carried out directly but need a limiting process for finding the solution.

Case and Moelius [10,25] applied this idea to the task of finding fixed points in numberings. From the beginnings of recursion theory, it is known that acceptable numberings of all partial recursive functions admit fixed-points, that is, for every recursive function f there is an e with $\varphi_{f(e)} = \varphi_e$ (where φ_e denotes the e-th function in the acceptable numbering φ). This means in particular that one cannot modify the behaviour of programs systematically; all methods to modify the program of functions fail to have an effect on some program e [33]. In their work on this topic, Case and Moelius [10]

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noted that Kleene's original version [23] of the fixed point theorem – known as Kleene's Recursion Theorem (KRT) – is in many applications much more useful than Rogers' variant. Informally, Kleene's Recursion Theorem says that for every partial recursive function η with two inputs there is an index e such that, for all x, $\varphi_e(x) = \eta(e, x)$. Moelius [25] studied in detail which numberings ψ satisfy KRT and, if they do, how complex it is to obtain the fixed point e as above. The above mentioned work of Moelius motivated choosing KRT rather than the popular fixed point theorem as the basic notion for the present paper.

A numbering is a uniformly partially recursive listing of all partial recursive functions with one input. It depends on the numbering ψ whether it satisfies KRT and, in the case that it does, how complex is the process to find, for any given f, the fixed point e with $\psi_e = f(e, \cdot)$. The complexity of this process is measured in terms of the convergence criterion (finite, explanatory, vacillatory or behaviourally correct, see formal definitions in Section 2) used to find such a fixed point for all recursive functions f. These properties permit to classify numberings as those which obey FinKrt, ExKrt, VacKrt and BcKrt, respectively. Similarly one can measure the numberings ψ with respect to the question whether they are optimal for the corresponding learning criteria I; here ψ is optimal for I-learning iff every I-learnable class can be learnt using the numbering ψ as hypothesis space. A third way to classify numberings is to say that a numbering ψ is I-acceptable iff for every further numbering φ, one can translate each index e of φ into a sequence of ψ-indices which converge to ψ-indices for φ_e in the way prescribed by I.

Building on prior work of Case and Moelius [10,25], the present work relates three ways to categorise quality of numberings (degree of effectiveness of KRT, degree of optimality of the numbering for learning, degree of acceptability of the numbering) with each other. The results show that in general, one can find arbitrary combinations of the degrees of the effectiveness of KRT and the degrees of optimality of the numbering. Furthermore, for I being finite, explanatory or vacillatory convergence, numberings which are optimal for I learning and permit I-effective KRT are also I-acceptable. This means, that KRT and optimality are complementary independent properties which together give that the numbering is acceptable. This is now described more formally and precisely in the remaining part of the paper.

2. Technical definitions and overview

Recursion-theoretic concepts not covered below are treated as by Rogers [33]; the interested reader might also consult the books of Calude [5], Downey and Hirschfeldt [12], Nies [26], Odifreddi [27,28], Li and Vitányi [24] and Soare [35] for further background on recursion theory and Kolmogorov complexity. Furthermore, the interested reader is referred to the two editions of "Systems that Learn" (first edition by Osherson, Stob and Weinstein [29] and second by Jain, Osherson, Royer and Sharma [17]) for background on inductive inference.

Let $\mathbb{N}$ be the set of natural numbers, $\{0, 1, 2, \ldots\}$. Lowercase italicised letters range either over elements of $\mathbb{N}$ (such as a, b, c, d, e, i, j, k, m, n, x, y, z) or over functions (such as f, g, h); it is always clear from the context which of the two cases applies. Uppercase italicised letters (such as A, B, D, E, K) range over subsets of $\mathbb{N}$, unless stated otherwise. Lowercase Greek letters (such as α, β, γ, ψ, φ, ϕ, θ) range over partial functions from $\mathbb{N}$ to $\mathbb{N}$, unless stated otherwise.

For each non-empty $X \subseteq \mathbb{N}$, $\min X$ denotes the minimum element of X, where $\min \emptyset = \infty$. For each non-empty, finite $X \subseteq \mathbb{N}$, $\max X$ denotes the maximum element of X; furthermore, $\max \emptyset = -1$. The finite sets can be indexed by canonical indices where the e-th finite set D_e is chosen such that $D_0 = \emptyset$ and, for $e > 0$, D_e is the unique set satisfying $\sum_{d \in D_e} 2^d = e$.

Let $\langle \cdot, \cdot \rangle$ be Cantor's pairing function: $\langle x, y \rangle = (x+y)(x+y+1)/2 + y$. The pairing function is a recursive, order preserving bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ [33, page 64]; here order preserving means that $x \le x' \wedge y \le y' \Rightarrow \langle x, y \rangle \le \langle x', y' \rangle$. Note that $\langle 0, 0 \rangle = 0$ and, for each x and y, $\max\{x, y\} \le \langle x, y \rangle$.

For each one-argument partial function α and $x \in \mathbb{N}$, $\alpha(x)\downarrow$ denotes that $\alpha(x)$ converges; $\alpha(x)\uparrow$ denotes that $\alpha(x)$ diverges. So, for example, $\lambda x \centerdot \uparrow$ denotes the everywhere divergent partial function.

For each partial function f, $\mathrm{rng}(f)$ denotes the range of f. A *text* is a total (not necessarily recursive) function from $\mathbb{N}$ to $\mathbb{N} \cup \{\#\}$. For each text T and $i \in \mathbb{N}$, $T[i]$ denotes the initial segment of T of length i: $T(0)T(1)\ldots T(i-1)$. SEQ denotes the set of all finite initial segments of all texts. For each text T and partial function α, T is a text *for* α iff $\mathrm{rng}(T) - \{\#\}$ is the graph of α as coded using the pairing function $\langle \cdot, \cdot \rangle$:

$$\mathrm{rng}(T) - \{\#\} = \{\langle x, y \rangle : \alpha(x) = y \ \wedge \ x, y \in \mathbb{N}\}. \tag{1}$$

For a total function f, one often identifies f with its canonical text, that is, the text T with $T(i) = \langle i, f(i) \rangle$. Thus, $f[n]$ represents the initial segment of length n of this canonical text.

Let $\mathcal{S}_{const} = \{f : (\exists c)(\forall x)[f(x) = c]\}$ denote the class of all constant functions.

Let $\mathcal{P}$ be the collection of all partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$. For each $\psi \in \mathcal{P}$ and $p \in \mathbb{N}$, let ψ_p be shorthand for $\psi(\langle p, \cdot \rangle)$. An *effective numbering* of $\mathcal{P}$ is a $\psi \in \mathcal{P}$ such that

$$(\forall \alpha \in \mathcal{P})(\exists p \in \mathbb{N})[\psi_p = \alpha]. \tag{2}$$

The present work only deals with effective numberings of partial recursive functions; hence, for notational simplicity, the requirements "$(\forall \alpha \in \mathcal{P})(\exists p \in \mathbb{N})[\psi_p = \alpha]$" and that the "numbering is effective" are usually assumed without being explicitly postulated. For all ψ and p, s, if $x < s$ and $\psi_p(x)$ halts within s steps with output y, then let $\psi_{p,s}(x) = y$; else let $\psi_{p,s}(x) = \uparrow$. Note that while it is in general undecidable whether $\psi_p(x)$ is defined, it is decidable whether $\psi_{p,s}(x)$ is defined