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On the complexity of an expanded Tarski's fixed point problem under the componentwise ordering [☆]

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ABSTRACT

Let Π be a finite lattice of integer points in a box of $\mathbb{R}^n$ and f an increasing mapping in terms of the componentwise ordering from Π to itself. The well-known Tarski's fixed point theorem asserts that f has a fixed point in Π . A simple expansion of f from Π to a larger lattice C of integer points in a box of $\mathbb{R}^n$ yields that the smallest point in C is always a fixed point of f (an expanded Tarski's fixed point problem). By introducing an integer labeling rule and applying a cubic triangulation of the Euclidean space, we prove in this paper that the expanded Tarski's fixed point problem is in the class PPA when f is given as an oracle. It is shown in this paper that Nash equilibria of a bimatrix game can be reformulated as fixed points different from the smallest point in C of an increasing mapping from C to itself. This implies that the expanded Tarski's fixed point problem has at least the same complexity as that of the Nash equilibrium problem. As a byproduct, we also present a homotopy-like simplicial method to compute a Tarski fixed point of f . The method starts from an arbitrary lattice point and follows a finite simplicial path to a fixed point of f .

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1. Introduction

Let $\leq$ be a binary relation on a nonempty set S . The pair $(S, \leq)$ is a partially ordered set if $\leq$ is reflexive, transitive and antisymmetric. A lattice is a partially ordered set $(S, \leq)$, in which any two elements x and y have a least upper bound (supremum), $\sup_S(x, y) = \inf\{z \in S \mid x \leq z \text{ and } y \leq z\}$, and a greatest lower bound (infimum), $\inf_S(x, y) = \sup\{z \in S \mid z \leq x \text{ and } z \leq y\}$, in the set. A lattice $(S, \leq)$ is complete if every nonempty subset of S has a supremum and an infimum in S . Let f be a mapping from S to itself. We say f is an increasing mapping if $f(x) \leq f(y)$ for any x and y of S with $x \leq y$.

Theorem 1 ([32]). *If $(S, \leq)$ is a complete lattice and f an increasing mapping from S to itself, then there exists $x^* \in S$ such that $f(x^*) = x^*$, which is a fixed point of f .*

Note that Tarski's fixed point theorem is significantly different from the classical Brouwer, Sperner lemma, or Kakutani's fixed point theorem where the mapping f is assumed to be continuous or semi-continuous. Tarski's fixed point theorem

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does not deal with continuous functions, and it simply states that any order-preserving function on a complete lattice has a fixed point, and indeed a smallest fixed point and a largest fixed point. Tarski's fixed point theorem plays an important role in the development of supermodular games for economic analysis that is referred to the literature such as Fudenberg and Tirole [18], Milgrom and Roberts [25,26], Milgrom and Shannon [27], Topkis [34], and Vives [35]. Some other applications of Tarski's fixed point theorem can be found in Fleiner [16]. To prove that a supermodular game has a pure-strategy equilibrium, a common approach is to construct an increasing mapping from a complete lattice to itself and apply Tarski's fixed point theorem. When a supermodular game is formulated for economic analysis, the computation of equilibria or fixed points of an increasing mapping becomes an important issue (e.g., [14,15,34]).

Let $N = \{1, 2, \dots, n\}$ and $N_0 = \{1, 2, \dots, n+1\}$. For $x = (x_1, x_2, \dots, x_n)^\top$ and $y = (y_1, y_2, \dots, y_n)^\top$ of $\mathbb{R}^n$, $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, which is the componentwise ordering. Let $\Pi = \{x \in \mathbb{Z}^n \mid c \leq x \leq d\}$, where $c = (c_1, c_2, \dots, c_n)^\top$ and $d = (d_1, d_2, \dots, d_n)^\top$ are two finite integer points with $c < d$. In terms of the componentwise ordering, Π is a complete lattice. Let $f(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top$ be an increasing mapping from Π to itself. Tarski's fixed point theorem shows that f has a fixed point in Π . Let

$$C = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\},$$

where $a = (a_1, a_2, \dots, a_n)^\top = c - w$ and $b = (b_1, b_2, \dots, b_n)^\top = d$ with $w = (w_1, w_2, \dots, w_n)^\top \geq \mathbf{0}$ being any given nonzero integer vector. We expand f from Π to C by simply setting $f(x) = a$ for any $x \in C \setminus \Pi$. With this expansion, f becomes an increasing mapping from C to itself with $f(a) = a$ and consequently, we have an expanded Tarski's fixed point problem.

The problem we consider in this paper is the complexity of computing a fixed point of f in C different from a , which always exists as f has a fixed point in Π .

A preliminary study on the complexity of Tarski's fixed point theorem can be found in Chang et al. [3]. In this paper, we show that Tarski's fixed point theorem is in the class PPA (when f is given as an oracle).

As a subclass of total search problems, the PPA (Polynomial Parity Arguments on undirected graphs) class was proposed in Papadimitriou [28]. When graphs are required to be directed, PPA becomes PPAD (Polynomial Parity Arguments on directed graphs). It was shown in Papadimitriou [28] that the class PPAD contains the classical Brouwer and Sperner lemma fixed point problems. In the last two decades, tremendous efforts have been devoted to the class PPAD in the literature, which are referred to Chen et al. [4], Daskalakis et al. [9] and Goldberg [19] and the references therein. Recently, several PPA-complete results have been derived in Aisenberg et al. [1], Deng et al. [10,11] and the references therein. To prove our new result, i.e., the class PPA also contains the expanded Tarski fixed point problem, we introduce an integer labeling rule and apply a cubic triangulation of the Euclidean space. The work of this paper can be considered as an extension of that in Dang and Ye [8] and the idea is stimulated from the simplicial path-following approach to integer programming in Dang [6] and Dang and Maaren [7] and has its foundation in simplicial methods for computing fixed points of a continuous mapping that were pioneered in Scarf [29] and substantially developed in the literature (e.g., [2,5,12,13,21,22,24,30,33]).

To show that the expanded Tarski's fixed point problem has at least the same complexity as that of a Nash equilibrium problem, we reformulate Nash equilibria of a bimatrix game as fixed points different from the smallest point in C of an increasing mapping from C to itself. As a further contribution, based on the labeling rule and triangulation, we attain a homotopy-like simplicial method to compute a fixed point of f in Π . The method starts from an arbitrary lattice point and follows a finite simplicial path to a fixed point of f .

The rest of this paper is organized as follows. An integer labeling rule is introduced in Section 2. A polynomial-time reduction of the expanded Tarski's fixed point theorem to the class PPA is attained in Section 3. The reformulation of a bimatrix game as an expanded Tarski's fixed problem is presented in Section 4. A homotopy-like simplicial method is described in Section 5.

2. An integer labeling rule and its properties

For $y \in \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$, let $\Gamma(y, W)$ be the set given by

$$\Gamma(y, W) = \{t(x, 0) + (1-t)(y, 1) \mid x \in W \text{ and } 0 \leq t \leq 1\}.$$

An illustration of $\Gamma(y, W)$ can be found in Fig. 1 with $y = (0, 0)^\top$ and $W = \{x \in \mathbb{R}^2 \mid -2 \leq x_i \leq 2, i = 1, 2\}$. For $x \in C$, let $g(x) = (g_1(x), g_2(x), \dots, g_n(x))^\top = x - f(x)$. Let $x^0 = (x_1^0, x_2^0, \dots, x_n^0)^\top$ be a given integer point in C . To establish our PPA graph, we need to assign an integer label to each integer point in $\mathbb{Z}^n \times [-1, 0] \cup \Gamma(x^0, \mathbb{Z}^n)$, which is as follows.

Definition 1 (An integer labeling rule). For each point $(x, \gamma) \in \mathbb{Z}^n \times [-1, 0] \cup \Gamma(x^0, \mathbb{Z}^n)$ with $\gamma \in \{-1, 0, 1\}$, we assign to (x, γ) an integer label $\ell(x, \gamma) \in N_0 \cup \{0\}$ as follows:

1. $\ell(x^0, 1) = n + 1$.

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