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Mutual dimension and random sequences ☆

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ABSTRACT

If S and T are infinite sequences over a finite alphabet, then the *lower* and *upper mutual dimensions* $mdim(S : T)$ and $Mdim(S : T)$ are the upper and lower densities of the algorithmic information that is shared by S and T. In this paper we investigate the relationships between mutual dimension and *coupled randomness*, which is the algorithmic randomness of two sequences R_1 and R_2 with respect to probability measures that may be dependent on one another. For a restricted but interesting class of coupled probability measures we prove an explicit formula for the mutual dimensions $mdim(R_1 : R_2)$ and $Mdim(R_1 : R_2)$, and we show that the condition $Mdim(R_1 : R_2) = 0$ is necessary but not sufficient for R_1 and R_2 to be independently random.

We also identify conditions under which Billingsley generalizations of the mutual dimensions $mdim(S : T)$ and $Mdim(S : T)$ can be meaningfully defined; we show that under these conditions these generalized mutual dimensions have the "correct" relationships with the Billingsley generalizations of $dim(S)$, $Dim(S)$, $dim(T)$, and $Dim(T)$ that were developed and applied by Lutz and Mayordomo; and we prove a divergence formula for the values of these generalized mutual dimensions.

1. Introduction

Algorithmic information theory combines tools from the theory of computing and classical Shannon information theory to create new methods for quantifying information in an expanding variety of contexts. Two notable and related strengths of this approach that were evident from the beginning [11] are its abilities to quantify the information in and to assess the randomness of *individual* data objects.

Some useful mathematical objects, such as real numbers and execution traces of nonterminating processes, are intrinsically infinitary. The randomness of such objects was successfully defined very early [18] but it was only at the turn of the present century [15,14] that ideas of Hausdorff were reshaped in order to define *effective fractal dimensions*, which quantify the densities of algorithmic information in such infinitary objects. Effective fractal dimensions, of which there are now many, and their relations with randomness are now a significant part of algorithmic information theory [6].

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Many scientific challenges require us to quantify not only the information in an individual object, but also the information *shared* by two objects. The *mutual information* $I(X;Y)$ of classical Shannon information theory does something along these lines, but for two probability spaces of objects rather than for two individual objects [5]. The *algorithmic mutual information* $I(x:y)$, defined in terms of Kolmogorov complexity [13], quantifies the information shared by two individual finite objects x and y.

The present authors recently developed the *mutual dimensions* $mdim(x:y)$ and $Mdim(x:y)$ in order to quantify the density of algorithmic information shared by two infinitary objects x and y [4]. The objects x and y of interest in [4] are points in Euclidean spaces $\mathbb{R}^n$ and their images under computable functions, so the fine-scale geometry of $\mathbb{R}^n$ plays a major role there.

In this paper we investigate mutual dimensions further, with objectives that are more conventional in algorithmic information theory. Specifically, we focus on the lower and upper mutual dimensions $mdim(S:T)$ and $Mdim(S:T)$ between two sequences $S,T\in\Sigma^\infty$, where Σ is a finite alphabet. (If $\Sigma=\{0,1\}$, then we write $\mathbf{C}$ for the *Cantor space* Σ^∞.) The definitions of these mutual dimensions, which are somewhat simpler in Σ^∞ than in $\mathbb{R}^n$, are implicit in [4] and explicit in section 2 below.

Our main objective here is to investigate the relationships between mutual dimension and *coupled randomness*, which is the algorithmic randomness of two sequences R_1 and R_2 with respect to probability measures that may be dependent on one another. In section 3 below we formulate coupled randomness precisely, and we prove our main theorem, Theorem 3.8, which gives an explicit formula for $mdim(R_1:R_2)$ and $Mdim(R_1:R_2)$ in a restricted but interesting class of coupled probability measures. This theorem can be regarded as a "mutual version" of Theorem 7.7 of [15], which in turn is an algorithmic extension of a classical theorem of Eggleston [7,2]. We also show in section 3 that $Mdim(R_1:R_2)=0$ is a necessary, but not sufficient condition for two random sequences R_1 and R_2 to be independently random.

In 1960 Billingsley investigated generalizations of Hausdorff dimension in which the dimension itself is defined "through the lens of" a given probability measure [1,3]. Lutz and Mayordomo developed the effective Billingsley dimensions $dim^\nu(S)$ and $Dim^\nu(S)$, where ν is a probability measure on Σ^∞, and these have been useful in the algorithmic information theory of self-similar fractals [17,8].

In section 4 we investigate "Billingsley generalizations" $mdim^\nu(S:T)$ and $Mdim^\nu(S:T)$ of $mdim(S:T)$ and $Mdim(S:T)$, where ν is a probability measure on $\Sigma^\infty\times\Sigma^\infty$. These turn out to make sense only when S and T are *mutually normalizable*, which means that the normalizations implicit in the fact that these dimensions are *densities* of shared information are the same for S as for T. We prove that, when mutual normalizability is satisfied, the Billingsley mutual dimensions $mdim^\nu(S:T)$ and $Mdim^\nu(S:T)$ are well behaved. We also identify a sufficient condition for mutual normalizability, make some preliminary observations on when it holds, and prove a divergence formula, analogous to a theorem of [16], for computing the values of the Billingsley mutual dimensions in many cases.

2. Mutual dimension in Cantor spaces

In [4] the authors defined and investigated the mutual dimension between points in Euclidean space. The purpose of this section is to develop a similar framework for the mutual dimension between sequences.

For $k>1$, let $\Sigma=\{0,1,\dots k-1\}$ be our alphabet, Σ^* be the set of all strings over Σ, and s_i be the ith string in the standard enumeration of Σ^*. Also, let Σ^∞ denote the set of all k-ary sequences over Σ. For $S,T\in\Sigma^\infty$, the notation (S,T) represents the sequence in $(\Sigma\times\Sigma)^\infty$ obtained after pairing each symbol in S with the symbol in T located at the same position. For $S\in\Sigma^\infty$, let

$$\alpha_S=\sum_{i=0}^{\infty}S[i]k^{-(i+1)}\in[0,1]. \tag{1}$$

Informally, we say that α_S is the *real representation* of S. Note that, in this section, we often use the notation $S\upharpoonright r$ to mean the first $r\in\mathbb{N}$ symbols of a sequence S.

We begin by reviewing some definitions and theorems of algorithmic information theory. Although any "flavor" of Kolmogorov complexity suffices for our purposes here, the ability to concatenate programs without explicit coding makes the prefix Kolmogorov complexity most convenient for us. Accordingly, every Turing machine here is assumed to be a prefix machine, i.e., a Turing machine M that takes two input strings, a *program* $\pi\in\{0,1\}^*$ and a side information string $w\in\Sigma^*$, and has the property that, for each $w\in\Sigma^*$, the set of all $\pi\in\{0,1\}^*$ on which $M(\pi,w)$ halts is prefix-free.

Definition. The *conditional (prefix) Kolmogorov complexity* of $u\in\Sigma^*$ given $w\in\Sigma^*$ with respect to a Turing machine M is

$$K_M(u\mid w)=\min\{|\pi|\,\big|\,\pi\in\{0,1\}^*\text{ and }M(\pi,w)=u\}.$$

We define the *(prefix) Kolmogorov complexity* of $u\in\Sigma^*$ with respect to a Turing machine M by $K_M(u)=K_M(u\mid\lambda)$, where λ is the *empty string*. In general, we write $M(\pi)$ for $M(\pi,\lambda)$.