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Decidability and independence of conjugacy problems in finitely presented monoids

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ABSTRACT

There have been several attempts to extend the notion of conjugacy from groups to monoids. The aim of this paper is study the decidability and independence of conjugacy problems for three of these notions (which we will denote by $\sim_p$, $\sim_o$, and $\sim_c$) in certain classes of finitely presented monoids. We will show that in the class of polycyclic monoids, p-conjugacy is "almost" transitive, $\sim_c$ is strictly included in $\sim_p$, and the p- and c-conjugacy problems are decidable with linear complexity on a two-tape Turing Machine. For other classes of monoids, the situation is more complicated. We show that there exists a monoid M defined by a finite complete presentation such that the c-conjugacy problem for M is undecidable, and that for finitely presented monoids, the c-conjugacy problem and the word problem are independent, as are the c-conjugacy and p-conjugacy problems. On other hand, we show that for finitely presented monoids, the o-conjugacy problem is reducible to the c-conjugacy problem.

1. Introduction

The well-known notion of conjugacy from group theory can be extended to monoids in many different ways. The authors dealt with four notions of conjugacy in monoids in [1,2]. The present paper can be considered an extension of this work.

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Any generalization of the conjugacy relation to general monoids must avoid inverses. One of the possible formulations, promulgated by Lallement [20] for a free monoid M, was the following relation:

$$a \sim_p b \Leftrightarrow \exists_{u,v \in M}\ a = uv \text{ and } b = vu. \tag{1.1}$$

(Lallement credited the idea of the relation $\sim_p$ to Lyndon and Schützenberger [22].) If M is a free monoid, then $\sim_p$ is an equivalence relation on M [20, Corollary 5.2], and so it can be regarded as a conjugacy in M. In a general monoid M, the relation $\sim_p$ is reflexive and symmetric, but not transitive. The transitive closure $\sim_p^*$ of $\sim_p$ has been defined as a conjugacy relation in a general semigroup [13,16,18,19]. (If $a \sim_p b$ in a general monoid, we say that a and b are *primarily conjugate* [19], hence our subscript in $\sim_p$).

Another relation that can serve as a conjugacy in any monoid is defined as follows:

$$a \sim_o b \Leftrightarrow \exists_{g,h \in M}\ ag = gb \text{ and } bh = ha. \tag{1.2}$$

This relation was defined by Otto for monoids presented by finite Thue systems [30], but it is an equivalence relation in any monoid. Its drawback – as a candidate for a conjugacy for general monoids – is that it reduces to the universal relation $M \times M$ for any monoid M that has a zero.

To remedy the latter problem, three authors of the present paper introduced a new notion of conjugacy [2], which retains Otto's concept for monoids without zero, but does not reduce to $M \times M$ if M has a zero. The main idea was to restrict the set from which conjugators can be chosen. For a monoid M with zero and $a \in M \setminus \{0\}$, let $\mathbb{P}(a)$ be the set $\{g \in M : (\forall m \in M)\ mag = 0 \ \Rightarrow\ ma = 0\}$, and define $\mathbb{P}(0)$ to be $\{0\}$. If M has no zero, we agree that $\mathbb{P}(a) = M$, for every $a \in M$. Following [2], we define a relation $\sim_c$ on any monoid M by

$$a \sim_c b \Leftrightarrow \exists_{g \in \mathbb{P}(a)} \exists_{h \in \mathbb{P}(b)}\ ag = gb \text{ and } bh = ha. \tag{1.3}$$

The relation $\sim_c$ is an equivalence relation on an arbitrary monoid M. Moreover, if M is a monoid without zero, then $\sim_c = \sim_o$; and if M is a free monoid, then $\sim_c = \sim_o = \sim_p$. In the case when M has a zero, the conjugacy class of 0 with respect to $\sim_c$ is $\{0\}$. Throughout the paper we shall refer to $\sim_i$, where $i \in \{p, o, c\}$, as i-conjugacy.

The aim of this paper is to study the decidability and independence of the i-conjugacy problems in some classes of finitely presented monoids.

It is well-known that the conjugacy problem for finitely presented groups is undecidable; that is, there exists a finitely presented group for which the conjugacy problem is undecidable [27]. The relations $\sim_p$, $\sim_o$, and $\sim_c$ reduce to group conjugacy when a monoid is a group. It follows that the i-conjugacy problem, for $i \in \{p, o, c\}$, is also undecidable. However, it is of interest to study decidability of the i-conjugacy problems in particular classes of finitely presented monoids.

First, we consider the class of polycyclic monoids, which are finitely presented monoids with zero. The polycyclic monoids P_n, with $n \geq 2$, were first introduced by Nivat and Perrot [26], and later rediscovered by Cuntz in the context of the theory of C^*-algebras [11, Section 1]. (Within the theory of C^*-algebras, the polycyclic monoids are often referred to as Cuntz inverse semigroups.) The polycyclic monoids appear to be related to the idea of self-similarity [14]. For example, the polycyclic monoid P_2 can be represented by partial injective maps on the Cantor set: its two generators, p_1 and p_2, map, respectively, the left and right hand sides of the Cantor set, to the whole Cantor set. These monoids can also be characterized as the syntactic monoid of the restricted Dyck language on a set of cardinality n, that is, the language that consists of all correct bracket sequences of n types of brackets. The study of representations of the polycyclic monoids naturally connects with the study of its conjugacy relations [21,17]. In [21], the classification of the 'proper closed inverse submonoids' of P_n depends on the study of its conjugacy classes.

In Section 3, we characterize p-conjugacy and c-conjugacy in the polycyclic monoids, and conclude that $\sim_c \subset \sim_p$. (For sets A and B, we write $A \subset B$ if A is a proper subset of B.) We then show that the p-conjugacy and c-conjugacy problems are decidable for polycyclic monoids, and that, given words a and b, testing whether or not $a \sim_i b$, for $i \in \{p, c\}$, can be done linearly on the lengths of a and b on a two-tape Turing Machine. Note that in a polycyclic monoid P_n, the relation $\sim_o$ is universal since P_n has a zero.

These positive results obtained for polycyclic monoids concerning the decidability and complexity of the conjugacy problems cannot be extended to the general finitely presented monoids.

In Section 4, we study decidability results. In particular, we show that there exists a monoid M defined by a finite complete presentation such that the c-conjugacy problem for M is undecidable (Proposition 4.2).

In Section 5, we study independence results. The word problem for groups is undecidable [23,28,31]. However, for groups, the word problem is reducible to the conjugacy problem [30, page 225], hence if the conjugacy problem for a group G is decidable, then the word problem for G is also decidable. Therefore, the word problem and the conjugacy problem for groups are not independent. The situation for monoids is different. Osipova [29] has proved that for finitely presented monoids, the word problem, the p-conjugacy problem, and the o-conjugacy problem are pairwise independent. We show that for finitely presented monoids, the word problem and the c-conjugacy problem are independent (Theorem 5.2), and that the p-conjugacy problem and the c-conjugacy problem are also independent (Theorem 5.3). We also show that the