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Convex and isometric domination of (weak) dominating pair graphs

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ABSTRACT

A set D of vertices in a graph G is a dominating set if every vertex of G, which is not in D, has a neighbor in D. A set of vertices D in G is convex (respectively, isometric), if all vertices in all shortest paths (respectively, all vertices in one of the shortest paths) between any two vertices in D lie in D. The problem of finding a minimum convex dominating (respectively, isometric dominating) set is considered in this paper from algorithmic point of view. For the class of weak dominating pair graphs (i.e., the graphs that contain a dominating pair, which is a pair of vertices $x, y \in V(G)$ such that vertices of any path between x and y form a dominating set), we present an efficient algorithm that finds a minimum isometric dominating set of such a graph. On the other hand, we prove that even if one restricts to weak dominating pair graphs that are also chordal graphs, the problem of deciding whether there exists a convex dominating set bounded by a given arbitrary positive integer is NP-complete. By further restricting the class of graphs to chordal dominating pair graphs (i.e., the chordal graphs in which every connected induced subgraph has a dominating pair) we are able to find a polynomial time algorithm that determines the minimum size of a convex dominating set of such a graph.

1. Introduction

Domination theory is one of the classical and most studied topics of graph theory; it was surveyed in two monographs that were published almost twenty years ago [12,11], and the theory has been extensively developed also in the last two decades. While in the basic version of domination, a *dominating set* D is a set of vertices in a graph G such that any vertex of G not in D has some neighbor in D, many variations of this concept have been introduced. In particular, in the so-called *connected domination*, as introduced in [23], a dominating set D is required to induce a connected subgraph. The idea reflects the requirements of potential applications, where vertices in D represent locations/nodes of discrete network, in which monitoring devices monitor the nodes in their closed neighborhoods, and it is desirable that one can move between locations/nodes, which are in D, by passing only through location/nodes that are in D. In the more restrictive case in which the time of moving between different nodes in D is also important, one can require that some shortest path between any two nodes in D lies completely in D (representing the so-called *weakly convex* or *isometric domination*); or, even more restrictively, that any shortest path between any two nodes in D lies completely in D (which then yields the so-called *convex domination*).

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Two graph invariants appear in this context. The *convex domination number* of a graph G, $\gamma_{con}(G)$, is the minimum cardinality of a set $D \subseteq V(G)$ such that D is at the same time a dominating set and a convex set (recall that a set D is convex if for any two vertices $x, y \in D$ all shortest x, y-paths lie in D). The *isometric domination number* of a graph G, $\gamma_{iso}(G)$, is the minimum cardinality of a set $D \subseteq V(G)$ such that D is at the same time a dominating set and an isometric set, where the latter means that for any two vertices $x, y \in D$ there exists a shortest x, y-path that lies in D. The study of convex domination and of isometric domination (introduced under then name weak convex domination) was initiated in 2004 by Lemańska [17] and Raczek [21], and was further studied from different points of view in several papers [14,16,18,22]. Raczek proved that the decision versions of isometric and convex domination number of a graph are NP-complete, even for bipartite and split graphs [21] (and hence also for chordal graphs). In fact, determining these numbers in split graphs is easily seen to be equivalent to the Set Cover Problem, one of the fundamental NP-complete problems due to Karp [13], see also [10].

The algorithmic and complexity issues were investigated recently for several other convexity parameters [3,5,9]. The theory of convex sets in graphs and other discrete structures was surveyed in the monograph already in 1993 [24], and it encompasses several important results in metric graph theory. In this developed part of graph theory (see also a survey on metric graph theory and geometry [2]) it is common to use the word *isometric subgraph* for a distance-preserving subgraph, while weak convexity usually refers to some form of convexity related to vertices of small distance. From this reason we suggest the name *isometric domination* instead of *weak domination*.

It is natural to consider these concepts in classes of graphs in which one can easily find nontrivial dominating sets, which are at the same time convex or isometric sets (nontrivial in this case means that the sets are not equal to $V(G)$). In particular, it is easy to see that removing all simplicial vertices in a chordal graph G, yields a subset of $V(G)$, which is convex and dominating. As mentioned above, the exact convex domination number is hard in split graphs and hence also in chordal graphs. Another interesting class of graphs in this context is that of asteroidal-triple-free graphs (AT-free graphs, for short); these graphs are defined as the graphs containing no *asteroidal triples*, i.e. independent sets of three vertices such that each pair is joined by a path that avoids the neighborhood of the third. The class of AT-free graphs contains many known classes of graphs such as interval, permutation, trapezoid, and co-comparability graphs, which have interesting geometric representations, and have also been in the focus of algorithmic graph theory, e.g. see the monographs [4,19]. In [6] Corneil, Olariu and Stewart presented the evidence that the absence of asteroidal triples imposes linearity of the recognition of the mentioned four classes. They also proved that AT-free graphs contain a *dominating pair*, that is, a pair of vertices with the property that every path connecting them is a dominating set. A linear time algorithm to find a dominating pair in AT-free graphs was presented in [7].

More generally, a graph is called a *weak dominating pair graph* if it contains a dominating pair, while a graph is called a *dominating pair graph* if each of its connected induced subgraphs is a weak dominating pair graph. Both graph classes contain AT-free graphs, and were introduced by Deogun and Kratsch in [8], where also a characterization of chordal dominating pair graphs using forbidden induced subgraphs was proven. In [20] it was shown that chordal dominating pair graphs can be recognized in polynomial time.

In this paper, we prove that convex domination problem is NP-complete when restricted to chordal weak dominating pair graphs (see Section 3). On the other hand, we present in Section 4 a polynomial time algorithm to determine the convex domination number of an arbitrary chordal dominating pair graph. (As a corollary, the convex domination number of an interval graph can also be computed in polynomial time.) Finally, in Section 5 we give a polynomial time algorithm to determine the isometric domination number of a (weak) dominating pair graph in which a dominating pair is also given. (Since one can determine a dominating pair in AT-free graphs in polynomial time, the problem of isometric domination number is polynomial in these graphs.) We conclude the introduction by remarking that results in these paper demonstrate that complexity behaviour of convex and isometric domination problems can be significantly different; see Fig. 1 presenting the classes of graphs considered in this paper.

2. Preliminaries

All graphs considered in this paper are finite, simple, and undirected. The *neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while *neighborhood of a set* $X \subseteq V(G)$ is defined as $N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of a vertex $v \in V(G)$ is the set $N_G[v] = N(v) \cup \{v\}$, while *closed neighborhood of a set* $X \subseteq V(G)$ is defined as $N_G[X] = \bigcup_{v \in X} N_G[v]$. Given a set $X \subseteq V(G)$ and a vertex $u \in X$, we define $pn_G(u, X)$ as the set $\{w \in V(G) : N_G[w] \cap X = \{u\}\}$. A member of the set $pn_G(u, X)$ is said to be an *X-private neighbor of u in G*.

Let $X \subseteq V(G)$ be any subset of vertices of G. The subgraph of G induced by vertices of X will be denoted by $\langle X \rangle$. A *clique* of a graph G is a set $C \subseteq V(G)$ such that $\langle C \rangle$ is a complete graph. An *independent set* of a graph G is a set $I \subseteq V(G)$ such that no two vertices of I are adjacent.

A *dominating set* of a graph G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex from D. If X and Y are subsets of vertices in G, then X *dominates* Y in G if $Y \subseteq N_G[X]$.

A set $S \subseteq V(G)$ is a *convex set*, if for any two vertices $u, v \in S$ the set S contains all the vertices that lie on a shortest path between u and v. Given a set $T \subseteq V(G)$, the *convex hull* of T, denoted $CH(T)$, is the smallest convex set that contains T. It is obvious that $S = CH(S)$ if and only if S is a convex set. It is also easy to see that $T \subseteq S$ implies $CH(T) \subseteq CH(S)$.