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Ruling out FPT algorithms for Weighted Coloring on forests ☆,☆☆

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ABSTRACT

Given a graph G, a *proper k-coloring* of G is a partition $c = (S_i)_{i\in[0,k-1]}$ of $V(G)$ into k stable sets $S_0, \ldots, S_{k-1}$. Given a weight function $w : V(G) \to \mathbb{R}^+$, the *weight of a color* S_i is defined as $w(i) = \max_{v\in S_i} w(v)$ and the *weight of a coloring* c as $w(c) = \sum_{i=0}^{k-1} w(i)$. Guan and Zhu (1997) [11] defined the *weighted chromatic number* of a pair (G, w), denoted by $\sigma(G, w)$, as the minimum weight of a proper coloring of G. For a positive integer r, they also defined $\sigma(G, w; r)$ as the minimum of $w(c)$ among all proper r-colorings c of G.
The complexity of determining $\sigma(G, w)$ when G is a tree was open for almost 20 years, until Araújo et al. (2014) [1] recently proved that the problem cannot be solved in time $n^{o(\log n)}$ on n-vertex trees unless the Exponential Time Hypothesis (ETH) fails.
The objective of this article is to provide hardness results for computing $\sigma(G, w)$ and $\sigma(G, w; r)$ when G is a tree or a forest, relying on complexity assumptions weaker than the ETH. Namely, we study the problem from the viewpoint of parameterized complexity, and we assume the weaker hypothesis FPT $\neq$ W[1]. Building on the techniques of Araújo et al., we prove that when G is a forest, the decision problem of computing $\sigma(G, w)$ is W[1]-hard parameterized by the size of a largest connected component of G, and that computing $\sigma(G, w; r)$ is W[2]-hard parameterized by r. Our results rule out the existence of FPT algorithms for computing these invariants on trees or forests for many natural choices of the parameter.

1. Introduction

A *(vertex) k-coloring* of a graph $G = (V, E)$ is a function $c : V(G) \to \{0, \ldots, k-1\}$. Such coloring c is *proper* if $c(u) \neq c(v)$ for every edge $\{u, v\} \in E(G)$. All the colorings we consider in this paper are proper, hence we may omit the word "proper". The *chromatic number* $\chi(G)$ of G is the minimum integer k such that G admits a k-coloring. Given a graph G, determining $\chi(G)$ is the goal of the classical VERTEX COLORING problem. If c is a k-coloring of G, then $S_i = \{u \in V(G) \mid c(u) = i\}$ is a stable (a.k.a. independent) set. Consequently, a k-coloring c can be seen as a partition of $V(G)$ into stable sets $S_0, \ldots, S_{k-1}$. We often see a coloring as a partition in the sequel.

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We study a generalization of VERTEX COLORING for vertex-weighted graphs that has been defined by Guan and Zhu [11]. Given a graph G and a weight function $w : V(G) \to \mathbb{R}^+$, the *weight of a color* S_i is defined as $w(i) = \max_{v \in S_i} w(v)$. Then, the *weight of a coloring* c is $w(c) = \sum_{i=0}^{k-1} w(i)$. In the WEIGHTED COLORING problem, the goal is to determine the *weighted chromatic number* of a pair (G, w), denoted by $\sigma(G, w)$, which is the minimum weight of a coloring of (G, w). A coloring c of G such that $w(c) = \sigma(G, w)$ is an *optimal weighted coloring*. Guan and Zhu [11] also defined, for a positive integer r, $\sigma(G, w; r)$ as the minimum of $w(c)$ among all r-colorings c of G, or as $+\infty$ if no r-coloring exists. Note that $\sigma(G, w) = \min_{r \geq 1} \sigma(G, w; r)$. It is worth mentioning that the WEIGHTED COLORING problem is also sometimes called MAX-COLORING in the literature; see for instance [14,16].

Guan and Zhu defined this problem in order to study practical applications related to resource allocation, which they describe in detail in [11]. One should observe that if all the vertex weights are equal to one, then $\sigma(G, w) = \chi(G)$, for every graph G. Consequently, determining $\sigma(G, w)$ and $\sigma(G, w; r)$ are NP-hard problems on general graphs [13]. In fact, these problems have been shown to be NP-hard even on split graphs, interval graphs, triangle-free planar graphs with bounded degree, and bipartite graphs [5,6,10]. On the other hand, the weighted chromatic number of cographs and of some subclasses of bipartite graphs can be found in polynomial time [5,6].

In this work we focus on the case where G is a forest, which has attracted considerable attention in the literature. Concerning graphs of bounded treewidth,[1] Guan and Zhu [11] showed, by using standard dynamic programming techniques, that on an n-vertex graph of treewidth t the parameter $\sigma(G, w; r)$ can be computed in time

$$n^{O(r)} \cdot r^{O(t)}. \tag{1}$$

Guan and Zhu [11] left as an open problem whether WEIGHTED COLORING is polynomial on trees and, more generally, on graphs of bounded treewidth. Escoffier et al. [10] found a polynomial-time approximation scheme to solve WEIGHTED COLORING on bounded treewidth graphs, and Kavitha and Mestre [14] showed that the problem is in P on the class of trees where vertices with degree at least three induce a stable set.

But the question of Guan and Zhu has been answered only recently, when Araújo et al. [1] showed that, unless the Exponential Time Hypothesis (ETH)[2] fails, there is no algorithm computing the weighted chromatic number of n-vertex trees in time $n^{o(\log n)}$.

As discussed in [1], it is worth mentioning that the above lower bound is tight. Indeed, Guan and Zhu [11] showed that the maximum number of colors used by an optimal weighted coloring of any weighted graph (G, w) is at most its so-called *first-fit chromatic number* (see [11] for the definition), denoted by $\chi_{\mathsf{FF}}(G)$. On the other hand, Linhares and Reed [15] proved that for any n-vertex graph G of treewidth at most t, it holds that $\chi_{\mathsf{FF}}(G) = O(t \log n)$. These observations together with Equation (1) imply that the WEIGHTED COLORING problem can be solved on forests in time $n^{O(\log n)}$.

Therefore, WEIGHTED COLORING on forests is unlikely to be in P, as this would contradict the ETH, and also unlikely to be NP-hard, as in that case all problems in NP could be solved in subexponential time, contradicting again the ETH.

Our results. The objective of this article is to provide hardness results for computing $\sigma(G, w)$ and $\sigma(G, w; r)$ when G is a forest, relying on complexity assumptions weaker than the ETH. Namely, we study the problem from the viewpoint of parameterized complexity (the basic definitions can be found in Section 2), and we assume the weaker hypothesis $\mathsf{FPT} \neq \mathsf{W}[1]$. Indeed, it is well-known [4] that the ETH implies that $\mathsf{FPT} \neq \mathsf{W}[1]$, which in turn implies that $\mathsf{P} \neq \mathsf{NP}$.

Our first result is that when (G, w) is a weighted forest, the decision problem of computing $\sigma(G, w)$ is W[1]-hard parameterized by the size of a largest connected component of G. This is proved by a parameterized reduction from INDEPENDENT SET that builds on the techniques introduced by Araújo et al. [1]. This result essentially rules out the existence of FPT algorithms for WEIGHTED COLORING on forests for many natural choices of the parameter: cliquewidth, maximum degree, maximum diameter of a connected component, number of colors in an optimal weighted coloring, etc. Indeed, all these parameters are bounded by the size of a largest connected component of G (for the number of colors, this can be proved by using that they are bounded by $\chi_{\mathsf{FF}}(G)$ [11], which is easily seen to be bounded by the size of a largest connected component).

We then move our attention to the parameter $\sigma(G, w; r)$ and we prove, by a parameterized reduction from DOMINATING SET similar to the first one, that computing $\sigma(G, w; r)$ on forests is W[2]-hard parameterized by r. Interestingly, if we assume the ETH, our reduction together with the results of Chen et al. [3] stating that DOMINATING SET parameterized by the size of the solution cannot be solved in time $f(k) \cdot n^{o(k)}$ unless the ETH fails, imply that, on graphs of bounded treewidth, the running time given by Equation (1) is asymptotically optimal, that is, there is no algorithm computing $\sigma(G, w; r)$ on n-vertex graphs of bounded treewidth in time $n^{o(r)}$.

We would like to mention that, although our reductions use several key ideas introduced by Araújo et al. [1], our results are incomparable to those of [1].

As further research, it would be interesting to identify "reasonable" parameters yielding FPT algorithms for WEIGHTED COLORING on forests. Probably, it might make sense to consider combined parameters that take into account, on top of the aforementioned invariants, the number of distinct weights in the input weighted forest.

[1] We will not define treewidth here, just recall that forests are the graphs with treewidth 1; see [4,7].

[2] The ETH states that 3-SAT cannot be solved in subexponential time; see [12] for more details.