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Conditional (edge-)fault-tolerant strong Menger (edge) connectivity of folded hypercubes ☆

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ABSTRACT

Menger's theorem is a characterization of the connectivity in finite graphs in terms of the minimum number of disjoint paths that can be found between any pair of vertices. According to Menger's theorem, a graph G is k-connected if and only if any two vertices of G are connected by at least k internally disjoint paths. Moreover, there are at least $\kappa(G)$ internally disjoint paths and, at most, $\min\{\deg_G(u), \deg_G(v)\}$ internally disjoint paths between any two distinct vertices u, v in G. Motivated by this observation, Oh and Chen (resp., Qiao and Yang) proposed the (fault-tolerant) strong Menger (resp., edge) connectivity as follows.

A connected graph G is called strongly Menger (edge) connected if for any two distinct vertices x, y in G, there are $\min\{\deg_G(x), \deg_G(y)\}$ (edge-)disjoint paths between x and y. A graph G is called m-(edge-)fault-tolerant strongly Menger (edge) connected if $G - F$ remains strongly Menger (edge) connected for an arbitrary set $F \subseteq V(G)$ (resp., $F \subseteq E(G)$) with $|F| \le m$. A graph G is called m-conditional (edge-)fault-tolerant strongly Menger (edge) connected if $G - F$ remains strongly Menger (edge) connected for an arbitrary set $F \subseteq V(G)$ (resp., $F \subseteq E(G)$), $|F| \le m$ and $\delta(G - F) \ge 2$.

Qiao and Yang (2017) proved that all n-dimensional folded hypercubes are $(2n-2)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \ge 5$. Yang, Zhao and Zhang (2017) showed that all n-dimensional folded hypercubes are $(2n-3)$-conditional fault-tolerant strongly Menger connected for $n \ge 8$. In this paper, we improve the result of Qiao and Yang by showing that all n-dimensional folded hypercubes are $(3n-5)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \ge 5$. Moreover, we present an example to show that our result is optimal with respect to the maximum tolerated edge faults. In addition, we show that the result of Yang, Zhao and Zhang is optimal by proving that the n-dimensional folded hypercubes are not $(2n-2)$-conditional fault-tolerant strongly Menger connected for $n \ge 8$.

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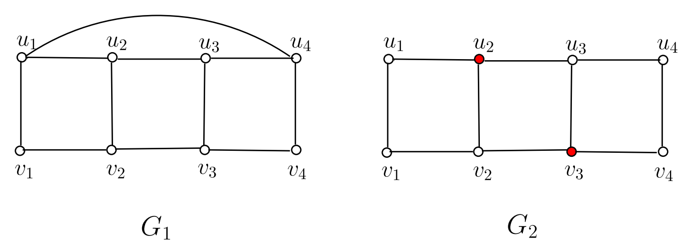

Fig. 1. Illustration of G_1 and G_2.

1. Introduction

The study of interconnection networks has been an important research area for parallel and distributed computer systems. They can be represented as graphs, where the vertices represent processors and the edges represent communication links.

For graph definitions and notations, we follow [1]. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices of G. For a set $F \subseteq E(G) \cup V(G)$, we use $G - F$ to denote the graph obtained by deleting F from G. The set of neighbors of a vertex u in G is denoted by $N_G(u)$, or briefly by $N(u)$. For a vertex set $U \subseteq V(G)$, the neighbors in $V(G) \setminus U$ of vertices in U are called neighbors of U, and can be denoted by $N(U)$. We use $\deg_G(u)$ to represent the number of neighbors of u in G. Moreover, we use $d_G(u, v)$ to represent the distance between u and v in G. For two disjoint subgraphs or vertex sets H_1, H_2 of G, we use $E(H_1, H_2)$ to denote the edges with one endpoint in H_1 and the other in H_2. For a given graph G, $x, y \in V(G)$, an x, y-path of length k is a finite sequence of distinct vertices $\langle v_0, v_1, \ldots v_k\rangle$ such that $x = v_0$, $y = v_k$, and $(v_i, v_{i+1}) \in E(G)$ for $0 \le i \le k - 1$. A set $F \subseteq V(G) \setminus \{x, y\}$ is an x, y-cut if $G - F$ has no x, y-path. Similarly, a set $F \subseteq E(G)$ is an x, y-edge cut if $G - F$ has no x, y-path.

In mathematics and computer science, (edge) connectivity is one of the basic concepts of graph theory. The vertex connectivity of G, namely $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or only has one vertex. The edge connectivity of G, namely $\lambda(G)$, is the minimum size of an edge set S such that $G - S$ is disconnected. Menger's theorem is a characterization of the (edge) connectivity in finite graphs in terms of the minimum number of (edge-)disjoint paths that can be found between any pair of vertices.

Theorem 1.1. [6] *(1) Let x and y be two distinct vertices of a graph G. For $(x, y) \notin E(G)$, the minimum size of an x, y-cut equals the maximum number of disjoint x, y-paths.*

(2) Let x and y be two distinct vertices of a graph G. The minimum size of an x, y-edge cut equals the maximum number of edge-disjoint x, y-paths.

It follows from this theorem that there are at least $\kappa(G)$ internally disjoint paths between any two distinct vertices u, v in G. Moreover, there are no more than $\min\{\deg_G(u), \deg_G(v)\}$ internally disjoint paths between u and v.

Let G_1 and G_2 be two distinct graphs (see Fig. 1), where
$V(G_1) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$;
$E(G_1) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (v_1, v_2), (v_2, v_3), (v_3, v_4), (u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4), (u_1, u_4)\}$;
$V(G_2) = V(G_1)$;
$E(G_2) = E(G_1) \setminus \{(u_1, u_4)\}$.

Obviously, $\kappa(G_1) = \kappa(G_2) = 2$. By observation, there are three internally disjoint paths between any two distinct vertices x, y in G_1, where $\deg_{G_1}(x) = \deg_{G_2}(y) = 3$. However, there are only two internally disjoint paths between u_2 and v_3 in G_2. In this case, G_1 is stronger than G_2. Motivated by this observation, Oh and Chen in [7] proposed strong Menger connectivity, which is also called maximal local-connectivity [2,3]. Similarly, Qiao and Yang [9] introduced strong Menger edge connectivity. For convenience, we redefine strong Menger (edge) connectivity as follows.

Definition 1.2. (1) A connected graph G is called strongly Menger connected if for any two distinct vertices x, y in G, there are $\min\{\deg_G(x), \deg_G(y)\}$ disjoint paths between x and y.

(2) A connected graph G is called strongly Menger edge connected if for any two distinct vertices x, y in G, there are $\min\{\deg_G(x), \deg_G(y)\}$ edge-disjoint paths between x and y.

Since interconnection network faults are unavoidable, fault-tolerance is quite important. Therefore, we need to consider fault-tolerance in real networks. Usually, we consider two kinds of fault-tolerant models in networks: random fault-tolerant models and conditional fault-tolerant models. Faults may occur anywhere, without restriction, in a random fault-tolerant model. For example, [2,3,5,8,11] mainly studied the random fault-tolerance. Next, we give the definition of (edge-)fault-tolerant strong Menger (edge) connectivity.