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On the reliability of alternating group graph-based networks

Yanze Huang [a], Limei Lin [b,*], Dajin Wang [c]

[a] School of Mathematics and Physics, Fujian Provincial Key Laboratory of Big Data Mining and Applications, Fujian University of Technology, Fuzhou, Fujian, 350118, PR China
[b] College of Mathematics and Informatics, Fujian Normal University, Fuzhou, Fujian, 350007, PR China
[c] Department of Computer Science, Montclair State University, Upper Montclair, 07043, NJ, USA

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ABSTRACT

The probability of having faults in a multiprocessor computer system increases as the size of system grows. One way to quantify the reliability of a system is using the probability that a fault-free subsystem of a certain size still exists with the presence of individual faults. The higher the probability is, the more reliable the system is. In this paper, we establish the reliability for networks based on AG_n, the n-dimensional alternating group graph. More specifically, we calculate the probability of a subnetwork (or subgraph) AG_n^{n-1} being fault-free, when given a single node's fault probability. Since subnetworks of AG_n intersect in highly complex manners, our scheme is to use the Principle of Inclusion–Exclusion to obtain a lower-bound of the probability, by considering intersections of up to four subgraphs. We show that the lower-bound derived this way is very close to the upper-bound obtained in a previous result, which means the lower-bound we get is a very tight one. Therefore, both lower-bound and upper-bound are close approximations of the accurate probability.

1. Introduction

The reliability of a system is the probability that the system has no faults. The processors in systems are connected through interconnection networks. Das and Kim [7] provided a unified task-based dependability model for hypercube computers. Soh et al. [15] improved lower bounds on the reliability of hypercube. Chang and Bhuyan [2] also proposed a combinatorial analysis of subcube reliability in hypercube under the probabilistic fault model. Kuo et al. [11] used various testing-efforts and fault-detection rates to propose a framework for modeling software reliability. In 2005, Chen et al. [4] developed a probabilistic approach to derive lower-bounds for the probability of hypercube network fault tolerance. Zhu et al. [21] proposed the reliability of folded hypercubes. Chen et al. [5] studied the probabilistic analysis on mesh network fault tolerance. In 2008, Wu and Latifi [19] derived an $(n-1)$-substar reliability using the probabilistic fault model. In 2012, Wang et al. [18] established a fault tolerance analysis of mesh networks with node failure probability. Liang et al. [13] studied the upper bounds on the connection probability for 2-D meshes and tori. In 2015, we [14] obtained an upper-bound reliability of subgraphs in the arrangement graph. In 2016, Li et al. [12] also established a reliability analysis of (n, k)-star graphs. The above studies did not propose the reliability of the alternating group graph (AG graph for short, AG_n for n-dimensional AG graph) [10] under the probabilistic fault model.

* Corresponding author.
E-mail address: putianlinlimei@163.com (L. Lin).

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The AG graph is an interconnection network for multicomputer systems. It has been shown to possess many attractive fault tolerant properties, including cycle-embedding [3,17], hamiltonicity [9], vertex pancyclicity [16]. Chiang and Chen [6] showed that the AG_n is isomorphic to arrangement graph $A_{n,n-2}$, making AG_n a special case of the arrangement graph [8].

In this paper, we use the probabilistic fault model to establish the reliability for the subgraphs of AG_n. A lower-bound analyzing method for calculating the probability of a subgraph AG_n^{n-1} being fault-free will be proposed, where a single-node's fault probability is given. We will show that the calculated lower-bound is very close to the upper-bound obtained from a previous result [14], which means that the lower-bound we get is a very tight one, and both lower- and upper-bound are very close approximations of the accurate probability.

The rest of this paper is organized as follows. Section 2 introduces some notations used throughout the paper and basic properties of the AG graph. Section 3 summarizes the previous result, and derives the upper-bound of $R_n^{n-1}(p)$ and the approximate $R_n^{n-1}(p)$. Section 4 presents our main result, which establishes a lower-bound on the reliability for AG_n^{n-1}, a large subgraph of the AG_n. Section 5 points out that the calculated lower-bound is very tight compared with the upper-bound, which is derived from the upper-bound of the arrangement graph obtained in a previous work [14]. Then the comparison result and its implication are discussed. Section 6 summarizes the paper with concluding remarks.

2. Preliminaries

Many large-scale multiprocessor computer systems take interconnection networks as underlying topologies, and an interconnection network is usually represented by a graph $G = G(V(G), E(G))$, where the node-set $V(G)$ is the set of processors and the edge-set $E(G)$ is the set of links. For notations and terminologies not defined here, please refer to paper [20]. The alternating group graph was first proposed by Jwo et al. [10] as an interconnection network topology. The definition of the alternating group graph is given as follows.

Let $\langle n\rangle = \{1, \ldots, n\}$ and $x = x_1x_2\ldots x_n$ where $x_i \in \langle n\rangle$ and $x_i \neq x_j$ for $i \neq j$. Two elements x_i and x_j is an inversion of x if $x_i < x_j$ for $i > j$. An even permutation is a permutation that contains an even number of inversions. Let Z_n be the set of all even permutations over $\langle n\rangle$. Let $s_i^- = (1i2)$ and $s_i^+ = (12i)$ for $i \in \{3, \ldots, n\}$ on Z_n by setting xs_i^- (resp., xs_i^+) to be the permutation obtained from x by rotating the symbols in positions 1, 2, and i from right to left (resp., left to right).

Definition 1. *[10] The n-dimensional alternating group graph, denoted by AG_n, is defined as follows:*

- *The node-set is $V(AG_n) = Z_n$.*
- *The edge-set is $E(AG_n) = \{xy \mid y = xs_i^- \text{ or } y = xs_i^+ \text{ for some } i \in \{3, \ldots, n\}\}$.*

Fig. 1 depicts the example of AG_4, where $Z_4 = \{1234, 1342, 1423, 2143, 2314, 2431, 3124, 3241, 3412, 4132, 4213, 4321\}$ and $s_3^- = (132) = \binom{1234}{3124}$, $s_3^+ = (123) = \binom{1234}{2314}$, $s_4^- = (142) = \binom{1234}{4132}$, $s_4^+ = (124) = \binom{1234}{2431}$. Let

$$V_n^{j:v} = \{x \mid x = x_1x_2\cdots x_{j-1}vx_{j+1}\cdots x_k \in E_n\}$$

for $j \in \{3, \ldots, n\}$ and $v \in \langle n\rangle$. $\{V_n^{j:v} \mid 1 \leq v \leq n\}$ forms a partition of $V(AG_n)$ for a fixed position $j \in \{3, \ldots, n\}$. Let $AG_n^{j:v}$ denote the subgraph of AG_n induced by $V_n^{j:v}$. Then $AG_n^{j:v}$ is isomorphic to AG_{n-1}. Thus, AG_n can be recursively constructed from n copies of subgraph AG_n^{n-1}. It is easy to check that each $AG_n^{j:v}$ is a subgraph of AG_n, and we say that AG_n can be decomposed into n copies of AG_n^{n-1} along the jth position.

We can fix m ($m > 1$) values at m positions to obtain smaller subgraphs of AG_n, and denote such an $(n-m)$-subalternating group graph of AG_n as AG_n^{n-m}. The number of *disjoint* AG_n^{n-m}s in an AG_n is $\binom{n}{m}m!$ for $1 \leq m \leq n-2$ and the number of *distinct* AG_n^{n-m}s is $\binom{n-2}{m}\binom{n}{m}m!$. Each subalternating group graph can be uniquely labeled as a string of symbols over the set $\{1, \ldots, n, X\}$, where the symbol X represents all unused digits.

Chiang and Chen [6] showed that the n-alternating group graph AG_n is isomorphic to $A_{n,n-2}$ [8]. The definition of the arrangement graph is given as follows.

Definition 2. *[8] Let $\langle n\rangle = \{1, \ldots, n\}$ and $P_{n,k}$ be a set of arrangements of k elements in $\langle n\rangle$ for $n > k$. The (n, k)-arrangement graph, denoted by $A_{n,k}$, is defined as follows:*

- *The node-set is $V(A_{n,k}) = P_{n,k}$.*
- *The edge-set is $E(A_{n,k}) = \{xy \mid x \text{ and } y \text{ differ in exactly one position}\}$.*

Let $R_n^{n-1}(p)$ be the probability that there exists a fault-free subgraph AG_n^{n-1} in an AG_n, where p is the probability that a node works. If $R_n^{n-1}(p)$ is high (low), then the probability that a operational subgraph of size $n-1$ exists is high (low).

We will first present the upper-bound for $R_n^{n-1}(p)$ and the approximation of $R_n^{n-1}(p)$ by using the known result on the (n, k)-arrangement graph [14]. Then we will establish a lower-bound for $R_n^{n-1}(p)$ using the probabilistic fault model and the *Principle of Inclusion–Exclusion (PIE)* [1].