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Polynomial-time algorithms for computing distances of fuzzy transition systems

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A B S T R A C T

Behaviour distances to measure the resemblance of two states in a (nondeterministic) fuzzy transition system have been proposed recently in literature. Such a distance, defined as a pseudo-ultrametric over the state space of the model, provides a quantitative analogue of bisimilarity. In this paper, we focus on the problem of computing these distances. We first extend the definition of the pseudo-ultrametric by introducing discount such that the discounting factor being equal to 1 captures the original definition. We then provide polynomial-time algorithms to calculate the behavioural distances, in both the non-discounted and the discounted setting. The algorithm is strongly polynomial in the former case.

1. Introduction

Fuzzy automata and fuzzy languages are standard computational devices for modelling uncertainty and imprecision due to fuzziness. Classical fuzzy automata are *deterministic*, namely, when in the current state and reading a symbol, the automaton can only move to a unique next (fuzzy) state. In [4], Cao et al. argued that *nondeterminism* is essential for modelling certain aspects of system, such as scheduling freedom, implementation freedom, the external environment, and incomplete information. Hence, they introduced nondeterminism into the model of fuzzy automata, giving rise to *nondeterministic fuzzy automata*, or more generally, (nondeterministic) *fuzzy transition systems*.

In general, system theory mainly concerns modelling systems and analysis of their properties. One of the fundamental questions studied in system theory is regarding the notion of equivalence, i.e., when can two systems be deemed the same and when can they be inter-substituted for each other? In the classical investigation in concurrency theory, *bisimulation*, introduced by Park and Milner [23], is a ubiquitous notion of equivalence which has become one of the primary tools in the analysis of systems: when two systems are bisimilar, known properties are readily transferred from one system to the other. However, it is now widely recognised that traditional equivalences are not a robust concept in the presence of *quantitative* (i.e. numerical) information in the model (see, e.g., [15]). Instead, it should come up with a more robust approach to distinguish system states. To accommodate this, researchers have borrowed from pure mathematics the notion of metric. A metric is often defined as a function that associates some *distance* with a pair of elements. Here, it is exploited to provide a measure of the discrepancy between two states that are not exactly bisimilar. Probabilistic systems and fuzzy transition

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systems are two typical examples of systems featuring quantitative nature. For probabilistic systems, the notion of distance in terms of pseudo-metrics has been studied extensively (cf. the related work). For fuzzy transition systems, Cao et al. [5] proposed a similar notion which serves as an analogue of those in probabilistic systems. Technically, a *pseudo-ultrametric*, instead of a pseudo-metric, was adopted. We refer the readers to Section 2 for formal definitions.

Having a proper definition of distance at hand, the next natural question is: how to compute it for a given pair of states? This raises some algorithmic challenges. For probabilistic systems, different algorithms have been provided for a variety of stochastic models (cf. the *related work*). However, to the best of our knowledge, little is known as for the corresponding algorithms in fuzzy transition systems. Indeed, in [5] this was left as an open problem, which is the main focus of the current paper.

On a different matter, *discounting* (or inflation) is a fundamental notion in economics and has been studied in, among others, Markov decision processes as well as game theory. Discounting represents the difference in importance between the future values and the present values. For instance, assuming a real-valued discount factor $0 < \gamma < 1$. A unit payoff is 1 if the payoff occurs today, but it becomes γ if it occurs tomorrow, γ^2 if it occurs the day after tomorrow, and so on. When $\gamma = 1$, the value is *not* discounted. Discounting has a natural place in system engineering; as a simple example, a potential bug in the far-away future is less troubling than a potential bug today [11]. In other words, discounting models preference for shorter solutions.

We introduce discounting into the distance definition for fuzzy transition systems, as done in probabilistic systems [28]. This is complementary to the definition given in [5]. In a nutshell, when measuring the distance between two states, the distance of their one-step successors are a times less important, and the distance between their two-step successors are a^2 times less important, etc.

Contributions. The main contributions of this paper are as follows:

(1) We extend the pseudo-ultrametric definition given in [5] for non-discounted setting to the *discounted* setting;
(2) We present polynomial-time algorithms to compute the behavioural distance, in both non-discounted (i.e., the original definition in [5]) and discounted setting (defined in the current paper).

Some explanations are in order. Regarding (1), remark that the definition in [5] is given in the *non-discounted* setting, where the present distances and the distances in future are equally weighted. In our setting, the discounting will be taken into consideration. Regarding (2), the basic ingredient of our algorithms is the standard "value iteration" procedure *à la* Kleene (Kleene's fixpoint theorem [22]). To qualify a polynomial-time algorithm, we show two facts: (i) For each iteration, it only needs polynomial time. Note that according to the definition of pseudo-ultrametric, each step requires to solve a (non-standard) mathematical programming problem (cf. Section 2). We show this can be done in polynomial-time. This part is identical for both discounted and non-discounted cases. (ii) The number of iterations is polynomially bounded. In the non-discounted case, this is done by inspecting the possible values appearing in each iteration. For the discounted case, unfortunately this does not hold. Instead, our strategy is to firstly compute an approximation of the sought value, and then apply the continued fraction algorithm to obtain the precise value. To the best of our knowledge, we are not aware of any previous work on polynomial algorithms for computing behaviour distances in fuzzy transition systems.

Fuzzy transition systems are known as *possibility systems* which are closely related to the *probabilistic* systems. Our algorithm and its analysis reveal some interesting difference between these two types of models, especially in the non-discounted case. Indeed, the scheme used in the paper cannot yield a polynomial-time algorithm for discrete-time Markov chains: There is an explicit example showing that it might take exponentially many iterations to reach the fixpoint; see [6]. As a matter of fact, for discrete-time Markov chains (which are the counterpart of deterministic fuzzy transition systems), polynomial-time algorithms do exist, but one has to appeal to linear programming [6]. This, however, does not provide a strongly polynomial-time algorithm.[1] Even worse, for Markov decision processes (which are the counterpart of nondeterministic fuzzy transition systems), the best known upper-bound is NP ∩ co-NP [18].[2] In contrast, here we give a strongly polynomial-time algorithm for (nondeterministic) fuzzy transition systems.

Related work.

Fuzzy systems, fuzzy automata and fuzzy transition systems. Conventionally, fuzzy systems are mainly referred to as fuzzy rule based systems where fuzzy states (outputs) evolve over time under some (maybe fuzzy) controls. In this paper, we are mainly interested in a type of fuzzy system models which are based on fuzzy automata [31]. Typically, fuzzy automata are considered to be acceptors of fuzzy languages. However, for the purpose of the current paper, we consider fuzzy transition systems, which are, in a nutshell, nondeterministic fuzzy automata without accepting states. Hence, we disregard the language aspect of fuzzy automata, but focus on their dynamics.

Metrics on other types of systems. Giacalone et al. [19] were the first to suggest a metric between *probabilistic transition systems* to formalise the notion of distance between processes. Subsequently, [15] studied a logical pseudometric for *labelled*

[1] It is a long-standing open problem whether linear programming admits a strongly polynomial-time algorithm.

[2] A (weakly) polynomial-time algorithm in this case would resolve a long-standing open problem on simple stochastic games for almost 30 years.