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Spy-game on graphs: Complexity and simple topologies ☆

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ABSTRACT

We define and study the following two-player game on a graph G. Let $k \in \mathbb{N}^*$. A set of k *guards* is occupying some vertices of G while one *spy* is standing at some node. At each turn, first the spy may move along at most s edges, where $s \in \mathbb{N}^*$ is his *speed*. Then, each guard may move along one edge. The spy and the guards may occupy the same vertices. The spy has to escape the surveillance of the guards, i.e., must reach a vertex at distance more than $d \in \mathbb{N}$ (a predefined distance) from every guard. Can the spy win against k guards? Similarly, what is the minimum distance d such that k guards may ensure that at least one of them remains at distance at most d from the spy? This game generalizes two well-studied games: Cops and robber games (when $s = 1$) and Eternal Dominating Set (when s is unbounded).

We consider the computational complexity of the problem, showing that it is NP-hard (for every speed s and distance d) and that some variant of it is PSPACE-hard in DAGs. Then, we establish tight tradeoffs between the number of guards, the speed s of the spy and the required distance d when G is a path or a cycle.

1. Introduction

Two-player turn-by-turn games on graphs have been well studied with two of the well-known games being the *Cops and robber game* [19,7] and the *eternal domination game* [14]. There exist many open problems in these two games and in this paper, we define and study a generalization of these games [9]. We believe that our game can lead to results of some of these open problems, much like other variants (see [6,13,2,8,11]) which have been studied for the same purpose. For example, a strategy for the guards in our game may be adapted to a strategy for the guards in the eternal domination game. As well, the motivation behind the study of the complexity of our game is that, as far as we know, it is not known whether the eternal domination game is PSPACE-hard (it is known to be NP-hard, but it is not known whether it is in NP).

We consider the following two-player game on a graph G, called *Spy-game*. Let $k, d, s \in \mathbb{N}$ be three integers such that $k > 0$ and $s > 0$. One player uses a set of k *guards* occupying some vertices of G while the other player plays with one *spy*

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initially standing at some node. This is a full information game, thus any player has full knowledge of the positions and previous moves of the other player. Note that several guards and even the spy could occupy the same vertex.

Initially, the spy is placed at some vertex of G. Then, the k guards are placed at some vertices of G. Then, the game proceeds turn-by-turn. At each turn, first the spy may move along at most s edges (s is the *speed* of the spy). Then, each guard may move along one edge. The spy wins if, after a finite number of turns (after the guards' move), it reaches a vertex at distance greater than d from every guard. The guards win otherwise, in which case we say that the guards *control* the spy at distance d, i.e. there is always at least one guard at distance at most d from the spy.

Given a graph G and two integers $d, s \in \mathbb{N}$, $s > 0$, let the *guard-number*, denoted by $gn_{s,d}(G)$, be the minimum number of guards required to control a spy with speed s at distance d, against all spy's strategies.

1.1. Preliminary remarks

We could define the game by placing the guards first. In that case, since the spy could choose its initial vertex at distance greater than d from any guard, we need to slightly modify the rules of the game. If the guards are placed first, they win if, after a finite number of turns, they ensure that the spy always remains at distance at most d from at least one guard. Equivalently, the spy wins if it can reach infinitely often a vertex at distance greater than d from every guard. We show that both versions of the game are closely related. In what follows, we consider the spy-game against a spy with speed s that must be controlled at distance d for some fixed integers $s > 0$ and d.

Claim 1. *If the spy wins against k guards in the game when it starts first, then it wins in the game when it is placed after the k guards.*

Proof of the claim. Assume that the spy has a winning strategy $\mathcal{S}$ when it is placed first. In particular, there is a vertex $v_0 \in V(G)$ such that, starting from v_0 and whatever be the strategy of the guards, the spy can reach a vertex at distance $> d$ from every guard. If the spy is placed after the guards, its strategy first consists in reaching v_0, and then in applying the strategy $\mathcal{S}$ until it is at distance $> d$ from every guard. The spy repeats this process infinitely often. □

The converse is not necessarily true, however we can prove a slightly weaker result which is actually tight. For this purpose, let us recall the definition of the well known *Cops and robber* game [19,7]. In this game, first k cops occupy some vertices of the graph. Then, one robber occupies a vertex. Turn-by-turn, each player may move its token (the cops first and then the robber) along an edge. The cops win if one of them reaches the same vertex as the robber after a finite number of turns. The robber wins otherwise. The *cop-number* $cn(G)$ of a graph G is the minimum number of cops required to win in G [1].

Claim 2. *If k guards win in the game when the spy is placed first in a graph G, then* $k + cn(G) - 1$ *guards win the game when they are placed first.*

Proof of the claim. Assume that k guards have a winning strategy when the spy is placed first. Such a strategy $\mathcal{S}$ is defined as follows. For any walk $W = (v_0, v_1, \cdots, v_\ell)$ of the spy,[1] each guard g_i $(1 \le i \le k)$ is assigned a vertex $pos_i(W)$, such that, for any vertex $w \in V(G)$ at distance at most s from v_ℓ and for any $i \le k$, $pos_i(W \cdot w) \in N[pos_i(W)]$ where $N[x]$ denote the set of vertices at distance at most one from $x \in V$. Moreover, for any walk $W = (v_0, \cdots, v_\ell)$ of the spy, there exists $i \le k$ such that the distance between v_ℓ and $pos_i(W)$ is at most d.

Now, let us assume that $k + cn(G) - 1$ guards are placed first. We show that after a finite number of turns, when the spy has followed any walk W, the vertices $pos_i(W)$ are occupied for all $1 \le i \le k$ and then the guards occupying these vertices can follow $\mathcal{S}$ and so win.

Let $0 \le j < k$ and assume that the spy has followed the walk $W = (v_0, \cdots, v_\ell)$ (in particular, the spy occupies v_ℓ) and that the vertices $pos_i(W)$ are occupied for all $1 \le i \le j$ ($j = 0$ means no such vertex is occupied). The guards occupying the vertices $pos_1(W), \cdots, pos_j(W)$ follow their strategy $\mathcal{S}$. There remains $k + cn(G) - 1 - j \ge cn(G)$ "free" guards. A team of $cn(G)$ of free guards will target the position $pos_{j+1}(W)$ (which acts as a robber moving at speed one in G). Therefore, after a finite number of steps, one free guard reaches $pos_{j+1}(W')$ (where W' is the walk that the spy has followed until this step). Continuing this way, after a finite number of steps, after that the spy has followed some walk W^*, the vertices $pos_i(W^*)$ are occupied for all $1 \le i \le k$. These k guards can go on executing $\mathcal{S}$ and win, which concludes the proof. □

The bound of the previous claim is tight. Indeed, for any graph G, $gn_{1,0}(G) = 1$ since one guard can be placed at the initial position of the spy and then follow it. On the other hand, if the guards are placed first, the game (for $s = 1$ and $d = 0$) is equivalent to the classical Cops and robber game and, therefore, $cn(G)$ guards are required.

[1] Here, a walk is a sequence of vertices (possibly with repetitions) such that two consecutive vertices in the sequence are at distance at most s.