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Minimum width color spanning annulus ☆

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ABSTRACT

Given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^2$ and each assigned with one of the given k distinct colors, we study the problem of finding the minimum width color spanning annulus of different shapes. Specifically, we consider the color spanning circular annulus ($CSCA$), axis-parallel square annulus ($CSSA$), axis parallel rectangular annulus ($CSRA$), and equilateral triangular annulus of fixed orientation ($CSETA$). The time complexities of the proposed algorithms for the respective problems are (i) $O(n^3 \log n)$ for $CSCA$, (ii) $O(n^3 + n^2 k \log k)$ for $CSSA$, (iii) $O(n^4)$ for $CSRA$, and (iv) $O(n^2 k)$ for $CSETA$.

1. Introduction

The motivation for studying color-spanning objects stems from the facility location problem. Here the input consists of different types of facilities, each having multiple copies, spread over a region. The objective is to identify a region of desired shape consisting of at least one copy of each facility and the measure of the region is optimized. In this paper, we study the minimum width color-spanning annulus problem of different shapes.

We are given a set $P = \{p_1, p_2, \ldots, p_n\}$ of points in $\mathbb{R}^2$, each colored with one of the given colors $\{1, 2, \ldots, k\}$, $k \leq n$, and for each of the colors $i \in \{1, 2, \ldots, k\}$ there exists at least one point of color i in P. A region is said to be *color-spanning* if it contains at least one point of each color in that region. An *annulus* $\mathcal{A}$ is a region bounded by two closed concentric geometric curves of same type, named as *inner boundary* C_{in} and *outer boundary* C_{out} respectively [10]. The (common) center c of C_{in} and C_{out} is referred to as the *annulus-center*, and the *width* of the annulus is the Euclidean distance between two closest points on the boundary of C_{in} and C_{out} respectively. A *color-spanning annulus* is an annulus that contains at least one point of each color. In this paper, we are interested in computing the color spanning circular annulus ($CSCA$), the color spanning axis-parallel square annulus ($CSSA$), the color spanning axis parallel rectangular annulus ($CSRA$), and the color-spanning equilateral triangular annulus of fixed orientation ($CSETA$), where the objective is to minimize the width of the corresponding annulus.

Related Work: The first (natural) variation of the color spanning facility location problem studied in the literature is the minimum radius color-spanning circle. The best known result is by Abellanas et al. [1,2]. Their proposed algorithm for computing the smallest color spanning circle runs in $O(kn \log n)$ time by using the technique of computing the upper envelope of Voronoi surfaces [15,19]. Abellanas et al. [1,2] also showed that the narrowest color-spanning strip and smallest axis-parallel color-spanning rectangle can be computed in $O(n^2 \alpha(k) \log k)$ and $O(n(n-k)\log^2 k)$ time respectively.

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Later Das et al. [12] improved the time complexity of narrowest color spanning strip problem to $O(n^2 \log n)$, and smallest color-spanning axis-parallel rectangle problem to $O(n(n-k)\log k)$. They provided a solution for the arbitrary oriented color-spanning rectangle problem in $O(n^3 \log k)$ time using $O(n)$ space. Recently, Khanteimouri et al. [16] presented an algorithm for color spanning square in $O(n \log^2 n)$ time, and Hasheminejad et al. [13] presented an algorithm for the color spanning axis-parallel equilateral triangle in $O(n \log n)$ time.

The problem of computing the minimum width annulus is also studied in the literature. Given a set of n points, computing the minimum width circular annulus containing all the points was independently addressed in [10,18,20]. All of these methods result in algorithms of time complexity $O(n^2)$. Sub-quadratic time algorithms for the circular annulus problem are also available in the literature. These are (i) an $O(n^{\frac{8}{5}+\epsilon})$ time deterministic algorithm proposed by Agarwal et al. [7] using parametric searching, and (ii) an $O(n^{\frac{3}{2}+\epsilon})$ time randomized algorithm proposed by Agarwal et al. [6]. To talk about the variations other than circular annulus for a general point set in $\mathbb{R}^2$, the well known results are an $O(n)$ time optimum algorithm for the axis parallel rectangular annulus by Abellanas et al. [3], and an $O(n \log n)$ time optimum algorithm for the axis parallel square annulus by Gluchshenko et al. [14]. Mukherjee et al. [17] proposed an algorithm for computing the minimum width axis parallel rectangular annulus for a point set in $\mathbb{R}^d$ in $O(nd)$ time. They also proposed an algorithm for arbitrary oriented minimum width rectangular annulus in $\mathbb{R}^2$ that runs in $O(n^2 \log n)$ time using $O(n)$ space (the definition of annulus in that paper [17] is slightly different from ours). Recently, Bae [8] proposed the minimum width square annulus of arbitrary orientation in $O(n^3 \log n)$ time.

Our Results: We first show that if the annulus-center is given, then the $CSCA$ problem can be solved in $\Theta(n \log n)$ time. Next, we propose an $O(n^2 \log n)$ time algorithm for the $CSCA$ problem where the annulus-center is constrained to lie on a given line. The unconstrained version of the $CSCA$ problem can be solved in $O(n^3 \log n)$ time. Next, we show that $CSSA$, $CSRA$ and $CSETA$ problems can be solved in $O(n^3 + n^2 k \log k)$, $O(n^4)$, and $O(n^2 k)$ time respectively. Each of these algorithms need $O(n)$ extra work-space. Similar methods work for computing the minimum width color-spanning circular annulus in L_1 norm [14] also. Observe that, if we change the norm of measuring the distance then the geometric characterizations for the minimum width circular annulus changes drastically. It remains an interesting open question to solve this problem in other norms. Finally, it needs to be mentioned that the minimum width annuli of all the considered shapes containing at least k monochromatic points can also be solved using the techniques for $CSCA$, $CSSA$, $CSRA$ and $CSETA$ problems respectively.

2. Preliminaries

Definition 1. An annulus containing a subset of points $P' \subseteq P$ is said to be *minimal* if it is of minimum width among all annuli that contain all the points in P'.

Definition 2. The interior of an annulus $\mathcal{A}$, defined by $INT(\mathcal{A})$, is the region inside $\mathcal{A}$ excluding C_{in} and C_{out}. A subset Π of points in P is said to *define* annulus $\mathcal{A}$ if these points lie on $C_{in} \cup C_{out}$ and uniquely define C_{in} and C_{out}.

Observation 1. (a) A subset Π of points defining a minimal color-spanning annulus $\mathcal{A}$ of a (multi-color) point set are of distinct colors, and (b) both C_{in} and C_{out} contain at least one point of Π, and (c) $INT(\mathcal{A})$ does not contain any point of color that defines the annulus $\mathcal{A}$.

Part (a) of Observation 1 follows from the fact that if more than one points of same color lie in Π, then we can reduce the width of $\mathcal{A}$ keeping one of them outside $\mathcal{A}$. Part (c) also follows from the same argument. Part (b) follows from the fact that if any one, say C_{in}, does not contain any point in Π, then we can further reduce the width of $\mathcal{A}$ by enlarging C_{in} until it touches a point inside $\mathcal{A}$. Note that, there may exist more than $|\Pi|$ points on $C_{in} \cup C_{out}$, but Π is necessary and sufficient to define $\mathcal{A}$ uniquely.

Throughout the paper, we use the following:

- The set of points of color θ is denoted as P_θ, and $n_\theta = |P_\theta|$, $\theta = 1, 2, \ldots, k$.
- The x- and y-coordinate of a point p are denoted as $x(p)$ and $y(p)$ respectively, and its color is denoted by $\chi(p)$.
- The distance between the pair of points $p_i, p_j \in P$ in L_k norm is denoted as $d_k(p_i, p_j)$.
- The distance of a line segment ℓ from a point p is the distance of a point $q \in \ell$ closest to p, and is denoted by $d_\perp(\ell, p)$.

2.1. Tight bounds for a constrained version

Here we analyze the time complexity of a constrained version of the *minimum width color-spanning $CSCA$* problem where the *annulus-center* is given.

Theorem 1. *The time complexity of computing the minimum width color spanning $CSCA$ around a given annulus-center c in any norm is $\Theta(n \log n)$.*