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Game comparison through play

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ABSTRACT

Absolute Universes of Combinatorial Games, as defined in a recent paper by the same authors, include many standard short Normal- Misère- and Scoring-play monoids. Given G and H in an Absolute Universe $\mathbb{U}$, we define a dual Normal-play game, called the Left Provisional Game $[G, H]$, and show that $G \succcurlyeq H$ if and only if Left wins $[G, H]$ playing second. As an example of our construction, we show how to compare Dicot Misère-play games in Siegel's computer program CGSuite and illustrate by including the partial order of all games of rank 2. We also show that Joyal's Normal-play Category generalizes to every Absolute Universe $\mathbb{U}$, and we define the associated categories **LNP**($\mathbb{U}$).

1. Introduction

This paper continues the work started in "Absolute Combinatorial Game Theory" [4] of short, two player games. Here we demonstrate that game comparison in an Absolute Universe can be interpreted as a dual Normal-play game (a player who cannot move loses). We anticipate a number of applications and we emphasise two. First, Absolute Universes of Combinatorial Games can be analyzed in CGSuite[3] and we illustrate by supplying partial order and code for Dicot Misère-play. Secondly, Joyal's Category for Normal-play games [3] generalizes to any Absolute Universe.

For a given winning convention (with 2 players Left and Right), game comparison boils down to: Left prefers G to H if, for all games X, Left does at least as well in $G + X$ as in $H + X$. ($G + X$ means that the current player plays in either G or X). Each different winning convention, possibly coupled with other constraints, gives a different partial order.

One of the most elegant discoveries of Normal-play CGT [1], is that Left wins playing second in the game G if and only if $G \geq 0$. Since Normal-play games constitute a group structure, this leads to a constructive (subordinate) general game comparison: for games G, H, $G \geq H$ if and only if Left wins the game $G - H$ playing second. Thus we have a convenient conversion of 'Abstract game-comparison' to *Play game-comparison*.[4]

The authors recently demonstrated [4] that there is a set of properties that define Absolute Universes and together these properties reduce game comparisons to considering only a certain *Proviso*, and a *Common Normal Part* (Section 2,

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3 CGSuite is a computer program, coded by A. Siegel, for evaluating combinatorial games.
4 This type of game comparison was preceded by Milnor's "Positional games" in the 1950s, but his class of games is not Absolute, so we cannot use it here.

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Theorem 2.4 in this paper). Except for Normal-play, typically Absolute Universes only have a monoid structure, so we cannot use the 'inverse' of any game freely.

For each Absolute Universe, we construct a dual Normal-play game, called the *Left Provisonal Game* (LPG), $[G, H]$ which is essentially playing $G - H$ (as if H were invertible) but where Left's options are restricted by the Proviso. By previous work [4] we show that the games G and H satisfy $G \succcurlyeq H$ if and only if Left wins the LPG $[G, H]$ whenever Right starts (Theorem 2.5).

By this, we can adapt existing results on Normal-play to any Absolute Universe, and we exemplify this by studying the Dicot Misère in Section 3 (with corresponding code in Appendix B). Section 4 concerns the second example on Category theory.

We give the relevant background on Absolute Combinatorial Game Theory [4] in Appendix A at the end of this paper.

2. Absolute game comparison and the Left Provisonal Game

First we recall the Proviso for a pair of games in a given Absolute Universe [4] (the relevant background on outcomes, left-atomic games and so on, is also given in Appendix A), and then we construct the dual Normal-play *Left Provisonal Game* (LPG).

Definition 2.1 *(Proviso).* Consider an Absolute Universe $\mathbb{U}$, and let $G, H \in \mathbb{U}$. The ordered pair of games $[G, H] \in \text{Proviso}(\mathbb{U}) \subseteq \mathbb{U} \times \mathbb{U}$ if

$o_L(G + X) \geqslant o_L(H + X)$ for all left-atomic games $X \in \mathbb{U}$;
$o_R(G + X) \geqslant o_R(H + X)$ for all right-atomic games $X \in \mathbb{U}$.

Ordered pairs of games in an Absolute Universe have a nice Normal-play game interpretation. That is, each ordered pair of games $[G, H] \in \mathbb{U} \times \mathbb{U}$ can be interpreted as a Normal-play game.

Definition 2.2 *(Left Provisonal Game).* Consider an Absolute Universe $\mathbb{U}$, and let $G, H \in \mathbb{U}$. The *Left Provisonal Game* $[G, H]$ is defined as follows:

(1) The Left options of $[G, H]$ are of the form:
  (a) $[G^L, H] \in \text{Proviso}(\mathbb{U})$, $G^L \in G^{\mathcal{L}}$;
  (b) $[G, H^R] \in \text{Proviso}(\mathbb{U})$, $H^R \in H^{\mathcal{R}}$.
(2) The Right options of $[G, H]$ are of the form $[G^R, H]$, $G^R \in G^{\mathcal{R}}$, and $[G, H^L]$, $H^L \in H^{\mathcal{L}}$;
(3) The player who, on their turn, cannot move loses.

Note that Right cannot move and loses playing first if both $G^{\mathcal{R}}$ and $H^{\mathcal{L}}$ are empty. For Left the situation is more intricate. If, for all $G^L \in G^{\mathcal{L}}$, $[G^L, H] \notin \text{Proviso}(\mathbb{U})$ and for all $H^R \in H^{\mathcal{R}}$, $[G, H^R] \notin \text{Proviso}(\mathbb{U})$, then Left cannot move and loses. Using the standard notation, thus $[G, H] = \{[G, H]^{\mathcal{L}} \mid [G, H]^{\mathcal{R}}\}$ is the Normal-play game

$$\{[G^L, H] \in \text{Proviso}(\mathbb{U}), [G, H^R] \in \text{Proviso}(\mathbb{U}) \mid [G^R, H], [G, H^L]\},$$

for all $G^L \in G^{\mathcal{L}}$, $H^R \in H^{\mathcal{R}}$, etc.

Definition 2.3 *(Left's maintenance).* Consider an Absolute Universe $\mathbb{U}$, and let $G, H \in \mathbb{U}$. The Left Provisonal Game $[G, H] \in \text{Maintain}(\mathbb{U})$ if, for all Right options $[G, H]^R \in [G, H]^{\mathcal{R}}$, there is a Left option $[G, H]^{RL}$, such that $[G, H]^{RL} \in \text{Maintain}(\mathbb{U})$.

Let us recall the main theorem for comparing games in an Absolute Universe, now stated as an equivalence involving Left Provisonal Games (also see Appendix A).

Theorem 2.4 *(Basic order of CGT, [4]). Consider an Absolute Universe $\mathbb{U}$ and let $G, H \in \mathbb{U}$. Then $G \succcurlyeq H$ if and only if $[G, H] \in$ Proviso$(\mathbb{U}) \cap$ Maintain$(\mathbb{U})$.*

Analogously:

Theorem 2.5. *Let G, H be games in an Absolute Universe $\mathbb{U}$. Then $G \succcurlyeq H$ if and only if $[G, H] \in$ Proviso$(\mathbb{U})$ and $[G, H] \geq 0$.*

Proof. By Theorem 2.4, it suffices to prove that $[G, H] \geq 0$ is equivalent with $[G, H] \in \text{Maintain}(\mathbb{U})$. This follows precisely because the inequality means Left wins playing second in Normal-play, which is Definition 2.2 (3) combined with the definition of Maintain$(\mathbb{U})$. □