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New infinite family of regular edge-isoperimetric graphs

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ABSTRACT

We introduce a new infinite family of regular graphs admitting nested solutions in the edge-isoperimetric problem for all their Cartesian powers. The obtained results include as special cases most of previously known results in this area.

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1. Introduction

Let $G = (V_G, E_G)$ be a graph and $A, B \subseteq V_G$. Denote

$$\begin{aligned} I_G(A, B) &= \{(u, v) \in E_G \mid u \in A,\ v \in B\}, \\ I_G(A) &= I_G(A, A), \\ I_G(m) &= \max_{A \subseteq V_G,\ |A|=m} |I_G(A)|. \end{aligned}$$

We will often omit the index G. Our subject is the following version of the *edge-isoperimetric problem* (EIP): for a fixed m, $1 \le m \le |V_G|$, find a set $A \subseteq V_G$ such that $|A| = m$ and $|I(A)| = I(m)$. We call such a set A *optimal*. This problem is known to be NP-complete in general and has many applications in various fields of knowledge, see survey [2].

We restrict ourselves to graphs representable as Cartesian products of other graphs. Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, their *Cartesian product* is defined as a graph $G \times H$ with the vertex-set $V_G \times V_H$ whose two vertices (x, y) and (u, v) are adjacent iff either $x = u$ and $(y, v) \in E_H$, or $(x, u) \in E_G$ and $y = v$. The graph $G^n = G \times G \times \cdots \times G$ (n times) is called the *nth Cartesian power* of G.

A particular interest in study of EIP is the case when there exists a total order $\mathcal{O}$ on the vertex set of graphs in question such that for every m the initial segment of this order of size m is an optimal set. Such order $\mathcal{O}$ is called *optimal order*. There exist graphs such that their Cartesian powers do not admit optimal orders. For example, it is known that there does not exist optimal orders for the second and higher powers of cycles of length p for $p > 5$ [6]. However, existence of a

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nested structure of solutions (that is, an optimal order) is an important graph property, because it provides as an immediate consequence solutions to many applied problems. Among such problems are the cutwidth, wirelength, and bisection width problems, construction of good k-partitioning of graphs and their embedding to some other graphs [2]. This stimulates the study of graphs which admit optimal orders for all their Cartesian powers. We call such graphs *edge-isoperimetric*.

The EIP for the Cartesian powers of a graph G has been well intensively studied for various graphs, see survey [2]. To summarize some of these results and present our new one we need to define the *lexicographic order* on a set of n-tuples with integer entries. For that we say that $(x_1, \dots, x_n)$ is greater than $(y_1, \dots, y_n)$ iff there exists an index i, $1 \le i \le n$, such that $x_j = y_j$ for $1 \le j < i$ and $x_i > y_i$. It turns out that just a few different optimal orders are discovered for the EIPs. Some of them are proved to work just for a few graphs [2,4]. This leaves the lexicographic order to work in most of known cases. In large this is due to the following result called in [1] *local-global principle*.

Theorem 1. (Ahlswede, Cai [1]) *If lexicographic order is optimal for $G \times G$ then it is optimal for G^n for any $n \ge 3$.*

The main difficulty in applying the local-global principle to a given graph G is to establish the optimality of the lexicographic order for $G \times G$. For this, however, no general methods have been developed so far. At this state of research a solution of the EIP on $G \times G$ for any other concrete graph G would be interesting and useful for developing more general methods. It seems difficult to characterize all graphs for whose all Cartesian powers the lexicographic order is optimal. Examples include graphs studied in [1,3,8,9]. However, we believe to the following conjecture.

Conjecture 1. *If lexicographic order is optimal for $G \times G$ then G is regular.*

All graphs studied in the above mentioned papers [1,3,8,9] are regular. In the light of this conjecture we emphasize on Cartesian powers of regular graphs. It is more convenient to work not directly with graphs, but with their numeric characteristic δ_G that we call *δ-sequence*. For a graph $G = (V, E)$ denote

$$\delta(m) = I(m) - I(m-1), \text{ with } \delta(1) = 0,$$
$$\delta_G = (\delta(1), \delta(2) \dots, \delta(|V|)).$$

For $|V_G| = p$ we call δ_G *symmetric* if

$$\delta(i) + \delta(p - i + 1) = \delta(p) \qquad \text{for } i = 1, \dots, p.$$

For example, the δ-sequence of the 3-dimensional unit cube $\delta_{Q^3} = (0, 1, 1, 2, 1, 2, 2, 3)$ is symmetric. It is easily shown that if G is regular then δ_G is symmetric. The recent result shows that the converse is also true.

Theorem 2. (Bonnet, Sikora [5]) *If δ_G is symmetric then G is regular.*

Our experience shows that in order for the lexicographic order to be optimal for $G \times G$, the graph G has to be dense, that is, have many edges. It seems that high density and regularity are crucial conditions for the lexicographic order to be optimal. Cliques have highest density and the lexicographic order is optimal for every their Cartesian power [9]. It is interesting to mention that removal of an edge from a clique leads to the next best choice for a high density graph, but products of this graph do not admit any optimal order. A natural way to construct dense regular graphs is to start with a clique K_p (or $K_{p,p}$) and remove a factor from it. In particular, one can consider removing $s \ge 1$ disjoint perfect matchings M from K_p or $K_{p,p}$. These are exactly the graphs being studied in [1,3,8,9]. The δ-sequences of these graphs are as follows:

- $\delta_{K_p} = (\{0, 1, 2, 3, \dots, p-1\})$
- $\delta_{K_p - sM} = (\{0, 1, 2, \dots, \frac{p}{2} - 1\},\ \{\frac{p}{2} - s, \frac{p}{2} - s + 1, \frac{p}{2} - s + 2, \dots, p - s - 1\})$
- $\delta_{K_{p,p}} = (\{0, 1\},\ \{1, 2\},\ \{2, 3\}, \dots, \{p-1, p\})$

The braces above show partitioning of δ-sequences into maximum monotonic subsequences. Thus, K_p has just one monotonic subsequence, $K_p - sM$ (a clique with s disjoint perfect matchings removed) has 2 (maximum) monotonic subsequences, and $K_{p,p}$ has p ones. Our main result generalizes all the above mentioned results for graphs $H_{s,p,i}$ defined by their δ-sequences. Namely, $\delta_{H_{s,p,i}}$ must admit partitioning into s monotonic subsequences of size p of the form:

$$(\{0, 1, \dots, p-1\},\ \{p-i, \dots, p-i+(p-1)\},\ \dots,\ \{(s-1)(p-i), \dots, (s-1)(p-i)+p-1\}).$$

For example, for $s = 3$, $p = 4$, $i = 2$ one has $\delta_{H_{3,4,2}} = (\{0, 1, 2, 3\},\ \{2, 3, 4, 5\},\ \{4, 5, 6, 7\})$.

In the next section we present a construction of some graphs $H_{s,p,i}$ with such δ-sequences. Our main result is as follows:

Theorem 3. (Main result) *Lexicographic order is optimal for $H_{s,p,i} \times H_{s,p,i}$ for $p \ge 3$, $s \ge 2$, and $1 \le i \le p - i$.*