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The switch Markov chain for sampling irregular graphs and digraphs ☆

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ABSTRACT

The problem of efficiently sampling from a set of (undirected, or directed) graphs with a given degree sequence has many applications. One approach to this problem uses a simple Markov chain, which we call the switch chain, to perform the sampling. The switch chain is known to be rapidly mixing for regular degree sequences, both in the undirected and directed setting.

We prove that the switch chain for undirected graphs is rapidly mixing for any degree sequence with minimum degree at least 1 and with maximum degree $d_{\max}$ which satisfies $3 \le d_{\max} \le \frac{1}{3}\sqrt{M}$, where M is the sum of the degrees. The mixing time bound obtained is only a factor n larger than that established in the regular case, where n is the number of vertices. Our result covers a wide range of degree sequences, including power-law density-bounded graphs with parameter $\gamma > 5/2$ and sufficiently many edges.

For directed degree sequences such that the switch chain is irreducible, we prove that the switch chain is rapidly mixing when all in-degrees and out-degrees are positive and bounded above by $\frac{1}{4}\sqrt{m}$, where m is the number of arcs, and not all in-degrees and out-degrees equal 1. The mixing time bound obtained in the directed case is an order of m^2 larger than that established in the regular case.

1. Introduction

There are several approaches to the problem of sampling from a set of graphs (or directed graphs) with a given degree sequence. In this paper we focus on the Markov chain approach. Here the running time of the sampling algorithm must be (deterministically) polynomially bounded but the output need not be exactly uniform: however, the user can specify how far from the uniform distribution the samples may be. Other approaches to the problem of sampling graphs (or directed graphs) are discussed in Section 1.1.

The switch chain is a natural and well-studied Markov chain for sampling from a set of graphs with a given degree sequence. Each move of the switch chain selects two distinct edges uniformly at random and attempts to replace these edges by a perfect matching of the four endvertices, chosen uniformly at random. The proposed move is rejected if the

☆ An earlier version of this work, for undirected graphs only, appeared in SODA 2015 [16].

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four endvertices are not distinct or if a multiple edge would be formed. We call each such move a *switch*. The precise definitions of the transitions for the switch chain for undirected and directed graphs are given at the start of Sections 2 and 3, respectively.

Ryser [34] used switches to study the structure of 0–1 matrices. Markov chains based on switches have been introduced by Besag and Clifford [5] for 0–1 matrices (bipartite graphs), Diaconis and Sturmfels [9] for contingency tables and Rao, Jana and Bandyopadhyay [33] for directed graphs.

The switch chain is aperiodic and its transition matrix is symmetric. It is well-known that the switch chain is irreducible for any (undirected) degree sequence: see [32,37]. Irreducibility for the directed chain is not guaranteed, see Rao et al. [33]. However, Berger and Müller-Hanneman [4] and LaMar [24,26] gave characterisations of directed degree sequences for which the switch chain is irreducible. In particular, the switch chain is irreducible for regular directed graphs (see for example Greenhill [15, Lemma 2.2]).

In order for the switch chain to be useful for sampling, it must converge quickly to its stationary distribution. The rate of convergence of a Markov chain $\mathcal{M}$ is captured by its *mixing time* $\tau(\mathcal{M}, \varepsilon)$, which is the minimum number of steps that the Markov chain $\mathcal{M}$ must run before its distribution is less than ε from stationarity, in total variation distance, from a worst-case starting state. A Markov chain with state space Ω is said to be *rapidly mixing* if its mixing time can be bounded above by some polynomial in $\log(|\Omega|)$ and $\log(\varepsilon^{-1})$. See Section 1.2 for more details.

Cooper, Dyer and Greenhill [7,8] showed that the switch chain is rapidly mixing for regular undirected graphs. Here the degree $d = d(n)$ may depend on n, the number of vertices. The mixing time bound is given as a polynomial in d and n. Earlier, Kannan, Tetali and Vempala [22] investigated the mixing time of the switch chain for regular bipartite graphs. Greenhill [15] proved that the switch chain for regular directed graphs (that is, d-in, d-out directed graphs) is rapidly mixing, again for any $d = d(n)$. Miklós, Erdős and Soukup [31] proved that the switch chain is rapidly mixing on half-regular bipartite graphs; that is, bipartite degree sequences which are regular for vertices on one side of the bipartition, but need not be regular for the other.

A multicommodity flow argument [35] was used in each of [7,15,22,31] to prove an upper bound on the mixing time of the switch chain. In each case, regularity (or half-regularity) was only required for one lemma, which we will call the *critical lemma*. This is a counting lemma which is used to bound the maximum load of the flow (see [7, Lemma 4], [15, Lemma 5.6] and [31, Lemma 6.15]).

In Section 2 we consider the undirected switch chain and prove the following theorem. This extends the rapid mixing result from [7] to irregular degree sequences which are not too dense.

Given a degree sequence $\boldsymbol{d} = (d_1, \ldots, d_n)$, write $\Omega(\boldsymbol{d})$ for the set of all (simple, undirected) graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ and degree sequence $\boldsymbol{d}$. Recall that $\boldsymbol{d}$ is called *graphical* when $\Omega(\boldsymbol{d})$ is nonempty. We restrict our attention to graphical sequences. Write $d_{\min}$ and $d_{\max}$ for the minimum and maximum degree in $\boldsymbol{d}$, respectively, and let $M = \sum_{j=1}^{n} d_j$ be the sum of the degrees.

Theorem 1.1. *Let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be a graphical degree sequence such that $d_{\min} \geq 1$ and $3 \leq d_{\max} \leq \frac{1}{3}\sqrt{M}$. The mixing time $\tau(\mathcal{M}(\boldsymbol{d}), \varepsilon)$ of the switch chain $\mathcal{M}(\boldsymbol{d})$ with state space $\Omega(\boldsymbol{d})$ satisfies*

$$\tau(\mathcal{M}(\boldsymbol{d}), \varepsilon) \leq d_{\max}^{14} M^9 \left(\tfrac{1}{2} M \log(M) + \log(\varepsilon^{-1})\right).$$

The proof of this result given in an earlier version of this paper [16] had a small gap in the proof. We have fixed the gap here, while also improving the upper bound on $d_{\max}$ by a small constant factor. However, we have not made a serious attempt to optimise the constants.

Theorem 1.1 covers many different degree sequences, for example:

- sparse graphs with constant average degree and maximum degree a sufficiently small constant times $\sqrt{n}$,
- dense graphs with linear average degree and maximum degree a sufficiently small constant times n,
- power-law density-bounded graphs with parameter $\gamma > 5/2$, when M is sufficiently large. Such graphs were considered by Gao and Wormald [13]: see in particular [13, Section 5], where they prove that $d_{\max} = O(M^{2/5})$ for such graphs (or in their notation, $\Delta = O(M_1^{2/5})$).

Since $M \leq d_{\max} n$, the mixing time bound given above is at most a factor of n larger than that obtained in [7,8] in the regular case.

The directed case is similar, and is considered in Section 3. To state our main result for the directed switch chain, we must introduce some notation. For definitions about directed graphs not given here, see [1].

A *directed degree sequence* is a sequence $\vec{\boldsymbol{d}}$ of ordered pairs of nonnegative integers $\vec{\boldsymbol{d}} = ((d_1^-, d_1^+), \ldots, (d_n^-, d_n^+))$, such that d_j^- is the in-degree and d_j^+ is the out-degree of vertex j, for all $j \in [n]$. (We use the arrow over the symbol $\vec{\boldsymbol{d}}$ so that our notation distinguishes directed and undirected degree sequences.) The directed degree sequence is *digraphical* if there exists a directed graph with these in-degrees and out-degrees. Write $\Omega(\vec{\boldsymbol{d}})$ for the set of all directed graphs with vertex set $[n]$