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Computing a minimum-width square annulus in arbitrary orientation ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we address the problem of computing a minimum-width square annulus in arbitrary orientation that encloses a given set of n points in the plane. A square annulus is the region between two concentric squares. We present an $O(n^3 \log n)$-time algorithm that finds such a square annulus over all orientations.

1. Introduction

An annulus informally depicts a ring-shaped region in the plane. More specifically, an annulus of a simple closed curve C, such as a circle, with a reference point inside C can be regarded as the region between two concentric homothets of C. Given a set P of n points in the plane, finding geometric shapes that best fit P is an important variant of shape matching problems. If the shape is restricted to C under a certain family of transformations, then this problem is equivalent to finding the *minimum-width annulus* that contains P. Among others, the case when C is chosen as a circle has been most intensively studied. The minimum-width circular annulus problem has been first addressed independently by Wainstein [17] and by Roy and Zhang [14], resulting in $O(n^2)$-time algorithms. The same time bound can be achieved by using the observation that the center of a minimum-width circular annulus corresponds to a vertex of the nearest-site Voronoi diagram of P, a vertex of the farthest-site Voronoi diagram of P, or an intersection point of two edges of the two diagrams [10]. The first sub-quadratic $O(n^{\frac{8}{5}+\epsilon})$-time algorithm was presented by Agarwal et al. [3] The currently best exact algorithm takes $O(n^{\frac{3}{2}+\epsilon})$ time by Agarwal and Sharir [2]. Linear-time approximation schemes are also known by Agarwal et al. [4] and by Chan [7].

The minimum-width circular annulus problem has applications in facility location in a sense that the center of the optimal annulus minimizes the difference between the maximum and the minimum distances from the center to input points with respect to the Euclidean metric. Of course, in some applications, other metrics like the L_1 or L_∞ metric would be more appropriate. In this sense, the *square annulus* or *rectangular annulus* problem naturally arises. Abellanas et al. [1] considered minimum-width rectangular annuli that are axis-parallel, and presented two algorithms taking $O(n)$ or $O(n \log n)$ time: one minimizes the width over rectangular annuli with arbitrary aspect ratio and the other does over rectangular annuli with a prescribed aspect ratio, respectively. Gluchshenko et al. [11] presented an $O(n \log n)$-time algorithm that computes a

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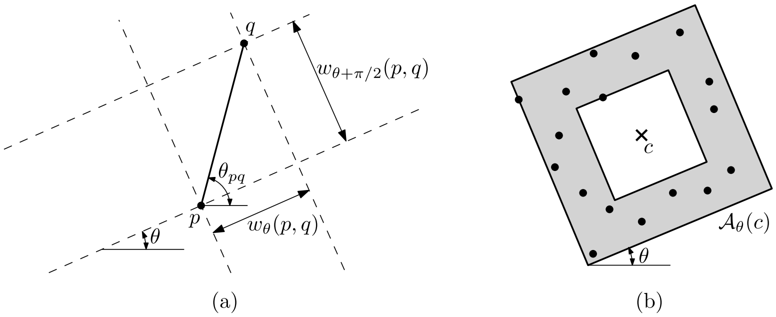

Fig. 1. (a) $w_\theta(p,q)$ and $w_{\theta+\pi/2}(p,q)$. (b) The minimum-width θ-aligned square annulus $\mathcal{A}_\theta(c)$ with center c enclosing points P.

minimum-width axis-parallel square annulus, and proved a matching lower bound, while the second algorithm by Abellanas et al. can do the same in the same time bound. The $\log n$ gap between the rectangular and the square annulus problems could be understood in a geometric point of view. In both cases, the outer boundary of an optimal annulus can be chosen as a smallest axis-parallel rectangle or square enclosing P, as shown in [1,11], but the smallest enclosing rectangle is unique while there are in general infinitely many smallest enclosing squares. If one considers rectangular or square annuli in arbitrary orientation, the problem gets more difficult. Mukherjee et al. [13] presented an $O(n^2 \log n)$-time algorithm that computes a minimum-width rectangular annulus in arbitrary orientation and arbitrary aspect ratio. However, to our best knowledge, there is no known algorithm for the minimum-width square annulus in arbitrary orientation. We aim to give the first algorithmic results to this variant of the problem.

A variant of the problem where the outer or inner boundary of the resulting annulus is fixed has also been studied. Duncan et al. [9] and De Berg et al. [8] independently showed that the minimum-width circular annulus can be computed in $O(n \log n)$ time in this case. Barequet et al. [5] and Barequet and Goryachev [6] considered the case when the prescribed shape C is given as any convex or simple polygon for this variant of the problem. When C is a square and its orientation can be chosen arbitrarily, their results imply that the minimum-width square annulus can be computed in $O(n^4 \log n)$ time, provided that the side length of its outer, inner or middle square is given.

In this work, we consider the minimum-width square annulus problem in arbitrary orientation, and present an $O(n^3 \log n)$-time exact algorithm. Note that this is the first algorithm for the problem. Comparing to the results of Barequet and Goryachev [6], our algorithm is more efficient while dropping the constraints on the size of the resulting annulus.

2. Preliminaries

For any square in the plane $\mathbb{R}^2$, its *center* is the intersection point of its two diagonals and its *radius* is half its side length. Two squares are called *concentric* if they share a common center and any pair of their sides are either parallel or orthogonal. A *square annulus* A is the region between two concentric squares, including its boundary. The *width* of a square annulus A is the difference of radii of the two concentric squares determining A.

The *orientation* of a line or line segment ℓ in the plane is a nonnegative value $\theta \in [0, \pi)$ such that the rotated copy of the x-axis by θ counter-clockwise is parallel to ℓ. If the orientation of a line or line segment is θ, then we say that the line or line segment is θ*-aligned*. A rectangle, a square, or a square annulus is also called θ-aligned for some $\theta \in [0, \pi/2)$ if each of its sides is either θ-aligned or $(\theta + \pi/2)$-aligned.

For any two points $p, q \in \mathbb{R}^2$, let $\overline{pq}$ denote the line segment joining p and q, and $|\overline{pq}|$ denote the Euclidean length of $\overline{pq}$. We will often discuss the distance between the orthogonal projections of p and q onto any θ-aligned line, denoted by $w_\theta(p,q)$. It is not difficult to see that $w_\theta(p,q) = |\overline{pq}| \cdot |\cos(\theta_{pq} - \theta)|$, where θ_{pq} denotes the orientation of $\overline{pq}$. See Fig. 1(a). Also, we define $d_\theta(p,q) := \max\{w_\theta(p,q), w_{\theta+\pi/2}(p,q)\}$ to be the convex distance between p and q with its unit disk being a unit θ-aligned square. Note that $d_\theta(p,q)$ is exactly the radius of the smallest θ-aligned square with center p that contains q in its boundary.

In a specific orientation $\theta \in [0, \pi/2)$, we regard any θ-aligned line to be *horizontal* and directed from left to right, and any $(\theta + \pi/2)$-aligned line to be *vertical* and directed from bottom to top. For any $p, q \in \mathbb{R}^2$, we say that p is to the *left* of q, or q is to the *right* of p, in θ if the orthogonal projection of p onto a θ-aligned line is prior to that of q. Similarly, p is *below* q or equivalently q is *above* p in θ if the orthogonal projection of p onto a $(\theta + \pi/2)$-aligned line is prior to that of q. For example, in Fig. 1(a), p is to the left of and below q in θ.

Let P be a set of points in $\mathbb{R}^2$. In orientation $\theta \in [0, \pi/2)$, let l^*_θ, r^*_θ, t^*_θ and b^*_θ be the leftmost, rightmost, topmost, and bottommost points in θ among those in P. Then, the smallest θ-aligned rectangle $\mathcal{R}_\theta$ enclosing P is uniquely determined by these four extreme points l^*_θ, r^*_θ, t^*_θ and b^*_θ. The *height* of $\mathcal{R}_\theta$ is the length of a vertical side of $\mathcal{R}_\theta$ and the *width* of $\mathcal{R}_\theta$ is the length of its horizontal side. That is, the height of $\mathcal{R}_\theta$ is equal to $w_{\theta+\pi/2}(t^*_\theta, b^*_\theta)$ and its width is equal to $w_\theta(l^*_\theta, r^*_\theta)$.

In this paper, we are interested in square annuli enclosing P. If we fix an orientation $\theta \in [0, \pi/2)$ and a center $c \in \mathbb{R}^2$, then there is a unique minimum-width θ-aligned square annulus containing P, which is determined by the smallest square that encloses P and the largest square whose interior contains no point of P. We denote this annulus by $\mathcal{A}_\theta(c)$. See Fig. 1(b).

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