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The many facets of upper domination

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A B S T R A C T

This paper studies Upper Domination, *i.e.*, the problem of computing the maximum cardinality of a minimal dominating set in a graph with respect to classical and parameterised complexity as well as approximability.

1. Introduction

A dominating set of an undirected graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that all vertices outside of S have a neighbour in S. The problem of finding the smallest dominating set of a given graph is one of the most widely studied problems in computational complexity. In this paper, we focus on a related problem that "flips" the optimisation objective. In Upper Domination we are given a graph and we are asked to find a *maximum cardinality* dominating set that is still minimal. A dominating set is minimal if any proper subset of it is no longer dominating, that is, if it does not contain obviously redundant vertices.

The study of MaxMin or MinMax versions of a problem by "flipping" the objective is not a new idea; in fact, such questions have been posed before for many classical optimisation problems. Some of the most well-known examples are the Minimum Maximal Independent Set problem [14,13,30,35] (also known as Minimum Independent Dominating Set), the Maximum Minimal Vertex Cover problem [11,45] and the Lazy Bureaucrat problem [4,8], which is a MinMax version of Knapsack. The initial motivation for this type of question was rather straightforward: most classical optimisation problems admit an easy, naive heuristic algorithm which starts with a trivial solution and then gradually tries to improve it in an

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obvious way until it gets stuck. For example, one can produce a (maximal) independent set of a graph by starting with a single vertex and then adding vertices to the current solution while maintaining an independent set. What can we say about the worst-case performance of such a basic algorithm? Motivated by this initial question the study of MaxMin and MinMax versions of standard optimisation problems has gradually grown into a sub-field with its own interest, often revealing new insights into the structure of the original problems. Upper Domination is a natural example of this family of problems and is also one of the six problems from the so-called *domination chain* (see [32] and Section 2), on which somewhat fewer results are currently known. The goal of this paper is to increase our understanding of this problem by investigating it from the different perspectives of approximability and classical and parameterised complexity.

1.1. Summary of results

We first link minimal dominating sets to a decomposition of the vertex set which turns out to be a useful tool throughout the whole paper.

From a classical complexity point of view, we show that Upper Domination is NP-hard on planar cubic graphs. Since the problem is easy on graphs of maximum degree 2, our results completely characterise the complexity of the problem in terms of maximum degree. Given the general behaviour of this type of problem, and the above results on Upper Domination in particular, the questions remains *why* are such problems typically so much harder than their original versions. Consider in this context the following *extension* problem: Given a graph $G = (V, E)$ and a set $S \subseteq V$, does there exist a minimal dominating set *of any size* that contains S? Even though questions of this type are typically trivial for problems such as Independent Set, we show that this kind of extension problem for Upper Domination is NP-hard even for planar cubic graphs. This helps explain the added difficulty of this problem, and more generally of problems of this type, since any natural algorithm that gradually builds a solution would have to contend with (some version of) this extension problem. On the positive side, we derive an exact $O^*(1.348^n)$-algorithm for subcubic graphs which builds on the decomposition derived in Section 2.

From the approximation perspective, we find that while Dominating Set admits a greedy $\ln n$ approximation, Upper Domination does not admit an $n^{1-\epsilon}$ approximation for any $\epsilon > 0$, unless P = NP. We also show that Upper Domination remains APX-hard on cubic graphs and complement these negative results by giving some approximation algorithms for the problem in restricted cases. Specifically, we show an $O(\Delta)$-approximation on graphs with maximum degree Δ, as well as an EPTAS on planar graphs.

From a parameterised point of view, we show that Upper Domination is W[1]-hard with respect to standard parameterisation (*i.e.* parameter $k = \Gamma(G)$, where $\Gamma(G)$ denotes the upper domination number). Conversely, Co-Upper Domination (*i.e.* Upper Domination with parameterisation $\ell = n - k$), is shown to be in FPT, which we prove by providing both a kernelisation and a branching algorithm.

2. Preliminaries and combinatorial bounds

We only deal with undirected simple connected graphs $G = (V, E)$. The number of vertices $n = |V|$ is known as the order of G. As usual, $N(v)$ denotes the open neighbourhood of v, and $N[v]$ is the closed neighbourhood of v, *i.e.*, $N[v] = N(v) \cup \{v\}$, which easily extends to vertex sets X, *i.e.*, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. The cardinality of $N(v)$ is known as the degree of v, denoted as $deg(v)$. The maximum degree in a graph is written as Δ. A graph of maximum degree 3 is called subcubic, and if all degrees equal 3, it is called cubic. In the area of parameterised and exact exponential algorithms, it has become customary not only to suppress constants (as in the O notation), but also polynomial-factors, leading to the so-called O^*-notation.

Given a graph $G = (V, E)$, a subset S of V is a *dominating set* if every vertex $v \in V \setminus S$ has at least one neighbour in S, *i.e.*, if $N[S] = V$. A dominating set is minimal if no proper subset of it is a dominating set. Likewise, a vertex set I is *independent* if $N(I) \cap I = \emptyset$. An independent set is maximal if no proper superset is independent. In the following we use classical notations: $\gamma(G)$ and $\Gamma(G)$ are the minimum and maximum cardinalities over all minimal dominating sets in G, $\alpha(G)$ and $i(G)$ are the maximum and minimum cardinalities over all maximal independent sets, and $\tau(G)$ is the size of a minimum vertex cover, which equals $|V| - \alpha(G)$ by Gallai's identity. A minimal dominating set D of G with $|D| = \Gamma(G)$ is also known as an *upper dominating set* of G, and $\Gamma(G)$ is also called the *upper domination number*.

For any subset $S \subseteq V$ and $v \in V$ we define the private neighbourhood of v with respect to S as $pn(v, S) := N[v] \setminus N[S \setminus \{v\}]$. Any $w \in pn(v, S)$ is called a *private neighbour of v with respect to S*. A set S is called *irredundant* if every vertex in it has at least one private neighbour, *i.e.*, if $|pn(v, S)| > 0$ for every $v \in S$. The cardinality of the largest irredundant set in G is denoted by $\mathrm{IR}(G)$, while $\mathrm{ir}(G)$ denotes the cardinality of the smallest maximal irredundant set in G. We can now observe the validity of the well-known *domination chain*:

$$\mathrm{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G)$$

The domination chain is largely due to the following two combinatorial properties: (1) Every maximal independent set is a minimal dominating set. (2) A dominating set $S \subseteq V$ is minimal if and only if $|pn(v, S)| > 0$ for every $v \in S$. Observe that v can be a private neighbour of itself, *i.e.*, a dominating set is minimal if and only if it is also an irredundant set. Actually, every minimal dominating set is also a maximal irredundant set.

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