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The switch operators and push-the-button games: A sequential compound over rulesets

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ABSTRACT

We study operators that combine combinatorial games. This field was initiated by Sprague–Grundy (1930s), Milnor (1950s) and Berlekamp–Conway–Guy (1970–80s) via the now classical disjunctive sum operator on (abstract) games. The new class consists in operators for *rulesets*, dubbed the *switch-operators*. The ordered pair of rulesets $(\mathcal{R}_1, \mathcal{R}_2)$ is *compatible* if, given any position in $\mathcal{R}_1$, there is a description of how to move in $\mathcal{R}_2$. Given compatible $(\mathcal{R}_1, \mathcal{R}_2)$, we build the *push-the-button* game $\mathcal{R}_1 \circledcirc \mathcal{R}_2$, where players start by playing according to the rules $\mathcal{R}_1$, but at some point during play, one of the players must *switch* the rules to $\mathcal{R}_2$, by pushing the button '$\circledcirc$'. Thus, the game ends according to the terminal condition of ruleset $\mathcal{R}_2$. We study the pairwise combinations of the classical rulesets Nim, Wythoff and Euclid. In addition, we prove that standard periodicity results for Subtraction games transfer to this setting, and we give partial results for a variation of Domineering, where $\mathcal{R}_1$ is the game where the players put the domino tiles horizontally and $\mathcal{R}_2$ the game where they play vertically (thus generalizing the octal game 0.07).

1. Gallimaufry – new combinations of games

Combinatorial Game Theory (CGT) concerns combinations of individual games. The most famous example is the *disjunctive game sum* operator, $G + H$, which, given two games G and H, consists in playing in either G or H until both games are exhausted. Several variations of this compound were defined by Conway in his reference book [10] and Berlekamp Conway Guy in [3]. One such example is the *ordinal sum* operator, which behaves like a disjunctive sum, with the additional constraint that any move in G annihilates the game H.

More recently, the so-called *sequential compound* operator was introduced [30]. It consists in playing successively the ordered pair of games (G, H), where H starts only when G is exhausted. These constructions are in this sense static: the starting position of the second game is fixed until the first game is exhausted. In this work, we study dynamic compounds, where the starting position of the second game depends on the moves that were made in the first game. As a consequence, our construction requires the full rulesets of the two games to build the compound.

In order to define such compounds, we will say that an ordered pair of combinatorial game rulesets $(\mathcal{R}_1, \mathcal{R}_2)$ is *compatible* if, given any position of $\mathcal{R}_1$, there is a description of how to move in $\mathcal{R}_2$. Note that we do not give a formal definition of this notion deliberately, as in the current context, we will only consider the case where $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same set

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of positions. Typically, a pair of rulesets $(\mathcal{R}_1, \mathcal{R}_2)$ is compatible whenever there is a function mapping any position of $\mathcal{R}_1$ into a position of $\mathcal{R}_2$. In our case, this function is just the identity. Given a compatible pair of rulesets $(\mathcal{R}_1, \mathcal{R}_2)$ and a specially designed *switch procedure* '$\Rightarrow$' (it can be a game in itself or anything else that declares a shift of rules), players start the game $\mathcal{R}_1 \Rightarrow \mathcal{R}_2$ with the rules $\mathcal{R}_1$. At their turn, a player can choose, instead of playing according to $\mathcal{R}_1$, to play in the switch procedure, and when this procedure has terminated, the rules are switched to $\mathcal{R}_2$ (using the current game configuration). Thus, we call this class of operators *switch operators* (or switch procedures).

In the current paper, the switch procedure is called the *push-the-button* operator (for short *push operator*). It consists in a button that must be pushed once and only once, by either player. Pushing the button counts as a move, hence after a player pushes the button, his opponent makes the next move. The button can be pushed any time, even before playing any move in $\mathcal{R}_1$, or when there is no move left in either game. In particular, if there is no move left in $\mathcal{R}_1$ and the button has not been pushed, then it has to be pushed before playing in $\mathcal{R}_2$ (the only way to invoke $\mathcal{R}_2$ is via the push-the-button move). In all cases, the game always ends according to the rules of $\mathcal{R}_2$.

We first recall some definitions and results about combinatorial game theory needed in this paper, and then give in Section 2 a formal definition of the push operator. In Section 3, we motivate this operator by the resolution of a particular game called Zeruclid, which we demonstrate is a push compound of the classical games of two-heap Nim [5] and Euclid [7]. Moreover we show that the second player's winning positions are similar to those of another classical game, Wythoff Nim (a.k.a Wythoff) [31]. Then, in Section 4 we continue by studying various push compounds of these three classical games. We also prove that standard periodicity results for Subtraction games transfer to this setting. We finish off by studying in Section 5 a variation of Domineering, where $\mathcal{R}_1$ is the game where the players put the domino tiles horizontally and $\mathcal{R}_2$ the game where they play vertically (thus generalizing the octal game 0.07).

1.1. More overview

Examples of popular traditional rulesets which can be considered as sequential compounds of several rulesets include Three Men's Morris, Picaria [24] (first place then slide to neighbors) and Nine men's morris (which is often played in three phases, first place, then slide and perhaps capture, and at last one of the players can move their pieces freely).

In addition, there has recently been scientific progress in building new games from combinations of the rules themselves. A nice illustration of this process is the game Clobbineering [6], whose rules are defined as a kind of union of the rules of the games Clobber and Domineering [2]. Indeed, in Clobbineering, the players choose either a Clobber or a Domineering move. Both types of moves are made on the same board.

One can also mention the more recent Building Nim [11], which behaves like Nine men's morris (i.e., the starting position of $\mathcal{R}_2$ is a final position of $\mathcal{R}_1$). However, if all these games are fun to play with, and may sometimes be solved thanks to ad hoc techniques, there is no known work relating to a formal definition of a compound of compatible rules.

A similar construction called *conjoined ruleset* has been recently introduced in [20] and used to create the games Go-Cut and Sno-Go which are compounds of the rulesets NoGo, Cut-Throat and Snort. In a conjoined ruleset, the players start playing according to a first ruleset until they reach a terminal position. Then, the game continues from the current position using the second ruleset. This case, as well as Building Nim, can be viewed as examples of a switch operator where the switch procedure automatically changes the rules to the second ruleset when the players reach a terminal position.

Like in a conjoined ruleset, the push-the-button operator can be considered as a *sequential compound of two sets of rules*, in the sense that the two compound sets are considered successively while playing. The main difference between these two switch procedures is that in the push-the-button operator, the rules can be changed at any time during the game. Moreover, one great interest of the push-the-button operator is its correlation with games allowing pass moves. Indeed, such games can be considered as a special instance of this operator, in analogy with how classical sequential compounds generalize the misère convention in combinatorial game theory [30].

1.2. Impartial games and their outcomes

The fundament of a combinatorial game is the set of game positions; a 'game board' together with some pieces (to place, move or remove etc.). The description of how to manipulate the pieces on the game board is the ruleset.[2]

Definition 1.2.1 *(Ruleset).* Given a set Ω of game positions, an impartial **ruleset** $\mathcal{R}$ over Ω is a function $\mathcal{R} : \Omega \to 2^{\Omega}$.

The function $\mathcal{R}$ gives the set of move options from each position in Ω. This ruleset can be extended to a function over sets of positions by saying that, for a set of positions $A \subset \Omega$, $\mathcal{R}(A) = \bigcup_{g \in A} \mathcal{R}(g)$.

Specifically, given a position $g \in \Omega$, the elements of $\mathcal{R}(g) \subset \Omega$ are called the *options* of g. Our study is carried out in the usual context of *short* games, i.e., games with a finite number of positions, and where no position is repeated during the play. The *followers* of $g \in \Omega$ are the positions in sets of the form $\mathcal{R}^i(g)$, where $\mathcal{R}^i = \mathcal{R}(\mathcal{R}^{i-1})$, for $i > 0$, and $\mathcal{R}^0$ is the

[2] Note that Definition 1.2.1 deals with impartial rulesets, i.e., rulesets where both players always have the same available moves during the play. However, it can be extended to partizan (i.e., non-impartial) games by separating the ruleset into two parts, each corresponding to the moves available to one player.