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One-to-one disjoint path covers in digraphs ☆

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A *one-to-one k-disjoint directed path cover* (*k-DDPC* for short) of a digraph D is a set of k disjoint directed paths joining source with sink that cover all the vertices of the digraph. Let $\delta^0(D) := min\{\delta^+(D), \delta^-(D)\}$ be the minimum semi-degree of D. We show that every digraph D of sufficiently large order n with $\delta^0(D) \geqslant \lceil (n+k+1)/2 \rceil$ contains a one-to-one k-DDPC for any given one distinct source-sink. The undirected version and a Ore-type degree conditions of this result was proved earlier [1].

1. Introduction

The *disjoint path cover* (*DPC* for short) problem arising in many areas such as software testing, database design and code optimization [2,3]. It have been investigated with respect to various special graphs such as hypercubes [4–6], recursive circulants [7–10], cubes of connected graphs [11,12], k-ary n-cubes [13,14] and spanning connected graphs [1]. Recently, Hyeong-Seok Lim, Hee-Chul Kim, and Jung-Heum Park [15] distinguish this problem into many situation such as many-to-many *k-DPC*, one-to-many *k-DPC*, paired or unpaired ones and so on.

One of the core subjects in Hamiltonian graph theory is to develop sufficient conditions for a graph to have a Hamiltonian path/cycle. Dirac [16] proved that a graph G of order $n \geq 3$ is Hamiltonian if its minimum degree $\delta \geqslant \frac{n}{2}$. A graph G is said to be Hamiltonian-connected if there is a Hamiltonian path between every two distinct vertices of G. Motivated by the DPC problem and Hamiltonian connectivity, we consider the degree condition of one-to-one k-DPC problem in digraph.

For basic notation not defined here, the reader could refer to [17]. We write $|S|$ as shorthand for $|V(S)|$ for any subset S of $V(D)$. Sometimes, we write $v \in D$, this means that $v \in V(D)$. When we say path, it always means directed path. When we say disjoint directed path cover, it always means that they don't have common vertex except endpoints.

Definition 1 *(One-to-one k-disjoint directed path cover).* A *one-to-one k-disjoint directed path link* of a digraph D is a set of k disjoint directed paths joining source to sink. If some one-to-one k-disjoint directed path link cover all vertices of the digraph, we call it a *one-to-one k-disjoint directed path cover* (*k-DDPC* for short).

Denote the source and sink by s_0 and t_0, respectively. Then the *one-to-one k-disjoint directed path link* joining s_0 and t_0 sometimes is called by *s_0-to-t_0 k-disjoint directed path link.*

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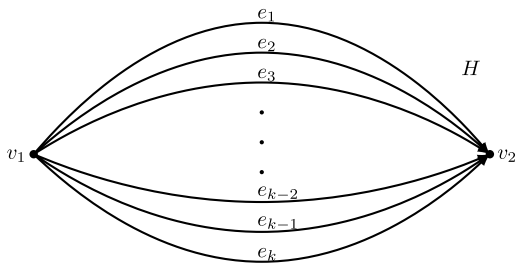

Fig. 1. There are k edges from v_1 to v_2 in H.

Definition 2. A digraph D is *one-to-one k-disjoint directed path coverable* if for any given distinct source s_0 and sink t_0, there exist a s_0*-to-*t_0 k-DDPC in D.

In this paper, we give the minimum semi-degree condition in the directed graph version of one-to-one *k-DPC* problem for every sufficiently large digraph D, more formally:

Theorem 1. *For every $k \geqslant 2$, there is an integer $n_0 = n_0(k)$ such that every digraph D on $n \geqslant n_0$ vertices with $\delta^0(D) \geqslant \lceil (n+k+1)/2 \rceil$ is one-to-one k-disjoint directed path coverable.*

Our proof shows that one can take $n_0(k) = Ck^9$ in which C is a sufficiently large constant.

2. Definitions and auxiliary facts

Let D_1 be a subgraph of D and vertex $a \in V(D)$. Let $N^+_{D_1}(a)$ denote the out neighbourhood of vertex a in $V(D_1) \setminus \{a\}$, i.e. $N^+_{D_1}(a) = \{v \in V(D_1) \setminus \{a\}\colon av \in E(D)\}$. We define $N^-_{D_1}(a)$ similarly. Moreover for any subset A of $V(D)$, we define $N^+_{D_1}(A)$ and $N^-_{D_1}(A)$ similarly. For any disjoint vertex set X and Y, if $x \in X$ and $y \in Y$, then the edge xy is an X–Y edge. The set of all X–Y edges is denoted by $E(X,Y)$. The number of edges in $E(X,Y)$ is denoted by $e(X,Y)$.

Theorem 2. *([18]) A digraph D of order n is Hamiltonian connected if $d^+_D(x) + d^-_D(y) \geqslant n+1$ whenever $xy \notin E(D)$.*

The next result of Michael Ferrara, Michael Jacobson and Florian Pfender [19] gives a sufficient condition for a digraph to be strongly k-connected.

A digraph D is H*-linked* if every injective mapping $f: V(H) \to V(D)$ can be extended to an H-subdivision in D.

Theorem 3. *([19]) Let H be a (multi)digraph. Digraph D is H-linked if $\delta^0(D) \geqslant \max\{\lceil \frac{1}{2}(|D| + \vec{b}_1(H) - 2)\rceil, 34|E(H)| + n_0(H)\}$, where $\vec{b}_1(H)$ denotes the maxima of $\vec{b}^1_{\mathcal{P}}(H) = |E(A \cup L_1 \cup L_2,\ B \cup L_1 \cup L_3)| + |E(B \cup R_1 \cup R_2,\ A \cup R_1 \cup R_3)| + |C| + \min\{|R| - |L_1|, |L| - |R_1|, |L_2| + |R_2|, |L_3| + |R_3|\}$ over all partition $\mathcal{P} = (A, B, C, L_1, L_2, L_3, R_1, R_2, R_3)$ of $V(H)$, $n_0(H)$ denotes the number of isolated vertices in H. This result is sharp whenever the first of the two bounds applies.*

Let H is a multidigraph and $|V(H)| = 2$, $|E(H)| = k$ (see Fig. 1).

In this case, we obtain $\vec{b}_1(H) = k$ by setting $L_2 = \{v_1\}$, $L_3 = \{v_2\}$. Note that digraph D is H-linked if and only if for every paired vertices $x, y \in D$, there are k disjoint paths from x to y in D, i.e. D is strongly k-connected. So we get the next corollary.

Corollary 4. *Let D be a digraph and $k \geqslant 2$. If $|D| \geqslant 80k$ and $\delta^0(D) \geqslant \lceil \frac{|D|+k}{2} \rceil - 1$, then digraph D is strongly k-connected.*

3. Preliminary results

Assume digraph D satisfy the hypotheses of Theorem 1. Corollary 4 implies the existence of an *one-to-one* k-disjoint path link in D. Let L be a maximum one-to-one k-disjoint path link (the k paths including as most vertices as possible) and suppose that L is not *one-to-one* k-DDPC. We denote the subdigraph of D induced by $V(D) \setminus V(L)$ by H. Our purpose is to find a longer *one-to-one k-disjoint path link* by modifying L (yielding a contradiction). In this section, we prove some lemmas and deduce some corollaries for preparation of our proof of Theorem 1.

Given $i \in \mathbb{N}$, let $F_i := \{x \in L : d^-_H(x) \geqslant i\}$, $T_i := \{x \in L : d^+_H(x) \geqslant i\}$ (let $F_1 = F$, $T_1 = T$). For a vertex x belongs to $V(P_i) \setminus \{s_0\}$, we define its previous vertex by x^-. There is a vertex $x \in V(P_i) \setminus \{t_0\}$, we define its next vertex by x^+. Let $F^-_i := \{x^- :$

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