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Drawing subcubic planar graphs with four slopes and optimal angular resolution ☆

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A B S T R A C T

A subcubic planar graph is a planar graph whose vertices have degree at most 3. We show that the subcubic planar graphs with at least five vertices have planar slope number at most 4, which is worst-case optimal. This answers an open question by Jelínek et al. [10]. Furthermore, we prove that the subcubic planar graphs with at least five vertices have angular resolution $\pi/4$, which solves an open problem by Kant [11] and by Formann et al. [8].

1. Introduction

A *straight-line drawing* of a graph G is a representation of G where the vertices are drawn as distinct points in the plane and the edges are drawn as line segments connecting the two corresponding end-points and not passing through any other point representing a vertex. Minimizing the number of slopes used by the edge segments of a straight-line drawing is a desirable aesthetic requirement and an interesting theoretical problem which has received considerable attention since its first definition by Wade and Chu [22].

The *slope number* of a graph G is defined as the minimum number of distinct slopes required by any straight-line drawing of G [22]. Let Δ be the maximum degree of a graph G and let m be the number of edges of G, then the slope number of G is at least $\frac{\Delta}{2}$ and at most m, as no more than two edges incident to the same vertex can have the same slope and at most one slope per edge can be used. For non-planar graphs, it has been shown that there exist graphs with $\Delta \geq 5$ whose slope number is unbounded (with respect to Δ) [1,20], while the maximum slope number of graphs with $\Delta = 4$ is still unknown, and the maximum slope numbers of graphs with $\Delta = 3$ is 4, which is worst-case optimal [18].

A *planar straight-line drawing* is a straight-line drawing that contains no edge crossings. The *planar slope number* of a planar graph G is defined as the minimum number of distinct slopes required by any planar straight-line drawing of G. Keszegh, Pach and Pálvölgyi proved that the planar slope number of a planar graph G is bounded by a function that is $2^{O(\Delta)}$ [13]; besides this upper bound, in the same paper a lower bound of $3\Delta - 6$, for $\Delta \geq 3$ is also proved [13]. The gap between these two bounds is large and the upper bound is probably far from being optimal, as pointed out by the authors themselves. Jelínek et al. study the *plane slope number* of plane partial 3-trees, i.e., planar partial 3-trees with a

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fixed planar embedding [10]. The plane slope number of an embedded planar graph G is the minimum number of distinct slopes required by any straight-line drawing of G that preserves the given embedding. Clearly the planar slope number is bounded from above by the plane slope number. Jelínek et al. proved that the plane slope number of any plane partial 3-tree with maximum degree Δ is at most $O(\Delta^5)$ [10]. Di Giacomo et al. studied a subclass of planar partial 3-trees with maximum degree Δ (those admitting an outer 1-planar drawing) and proved an $O(\Delta^2)$ upper bound for the planar slope number of these graphs [3]. Knauer, Micek and Walczak focus on a subclass of planar partial 3-trees, showing that the (outer)planar slope number of outerplanar graphs is at most $\Delta - 1$, for $\Delta \geq 4$, and this many slopes are sometimes needed for outerplanar drawings of outerplanar graphs [16]. Finally, Lenhart et al. proved that the planar slope number of a partial 2-tree is at most 2Δ and there exist partial 2-trees whose planar slope number is at least Δ, if Δ is odd, and at least $\Delta + 1$, if Δ is even [17]. Also, the plane slope number of a plane partial 2-tree of maximum degree Δ is at most 3Δ, and there exist plane 2-trees whose plane slope number is at least $3\Delta - 3$, if Δ is even, and at least $3\Delta - 4$, if Δ is odd [17].

Special interest has been devoted in the literature to the (not necessarily planar) slope number of (sub)cubic graphs, i.e., graphs having vertex degree (at most) 3. Keszegh et al. proved that the slope number of cubic graphs is 5 [14]. This result has been improved by Mukkamala and Szegedy who proved that the slope number of connected cubic graphs is 4 [19]. Finally, Mukkamala and Pálvölgyi showed that the four slopes $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ suffice for every cubic graph [18]. Concerning the planar slope number, Kant and independently Dujmović et al. proved that cubic 3-connected planar graphs have planar slope number 3 disregarding the slopes of three edges on the outerface [7,11]. Jelínek et al. showed that subcubic series-parallel graphs have planar slope number 3, which is worst-case optimal [9]. Jelínek et al. also asked to prove an upper bound on the planar slope number of subcubic planar graphs analogous to those in [15,18,19]. We answer this question as follows.

Theorem 1. *Let G be a subcubic planar graph with $n \geq 5$ vertices. The planar slope number of G is at most 4 and this bound is tight.*

Note that for $n \leq 4$ four slopes are not sufficient in general, since six slopes are necessary and sufficient for K_4. On the other hand, each subcubic planar graph with $n \leq 4$ vertices is a subgraph of K_4 and therefore six slopes are sufficient. The proof of Theorem 1 is based on a $O(n)$-time algorithm that computes a planar straight-line drawing of a subcubic planar graph using only slopes in the set $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. We assume the real RAM model of computation due to the fact that the geometric coordinates of the vertices may become arbitrarily large. Thus, another contribution of this paper is the following.

Theorem 2. *Every subcubic planar graph with $n \geq 5$ vertices has a straight-line planar drawing whose angular resolution is $\frac{\pi}{4}$, which is worst-case optimal. The drawing can be computed in $O(n)$ time in the real RAM model of computation.*

About Theorem 2, we recall that Formann et al. initiate the study of straight-line planar drawings with good angular resolution [8]. The *angular resolution* of a straight-line drawing is the minimum angle between two edges incident to a common vertex. Among the many questions that stimulated further research, Formann et al. ask whether every subcubic planar graph has a planar straight-line drawing such that the smallest angle is a constant independent of the size of the graph. An answer to this fundamental question has been already given in a paper by Kant, who claims that every subcubic planar graph with $n \geq 6$ has a planar straight-line drawing with all angles at least $\frac{\pi}{3}$ except for four angles which are at least $\frac{\pi}{6}$ [11]. This claim is correct if restricted to 3-connected subcubic planar graphs, but unfortunately incorrect in the general case as observed by Dujmović et al., who provided as a counter-example a family of connected subcubic planar graphs requiring a linear number of angles less than $\frac{\pi}{3}$ [7]. In [11], Kant also asks the following: Does every subcubic planar graph admit a straight-line planar drawing such that the smallest angle is at least $\frac{\pi}{4}$? Theorem 2 answers in the affirmative both the question by Formann et al. and the one by Kant.

The structure of our proofs is as follows. In Section 4 we prove Theorems 1 and 2 for simply 2-connected subcubic planar graphs. In Section 5 we extend the proof to simply connected subcubic planar graphs. In Section 6 we study the 3-connected subfamily. A basic tool for the proofs in the above mentioned sections is a drawing technique, given in Section 3, for *almost 3-connected* subcubic planar graphs, which are graphs that can be obtained from a 3-connected cubic planar graph by removing one edge.

In addition, in Section 6 we shall pay particular attention to the area of the drawings of the 3-connected cubic planar graphs, since this has been a topic of independent interest in the literature. Namely, we are going to show that a drawing that uses a grid of size of $(2n - 14) \times (2n - 14)$ can always be computed by using four slopes. We remark that a quadratic area bound for straight-line drawings of 3-connected cubic planar graphs was already proved by Kant [11], however using six slopes instead of four. Our result is also related to an open problem by Dujmović et al. [7]; they ask whether every 3-connected cubic plane graph has a plane drawing with polynomial area, such that every edge has one of three slopes except for three edges on the outerface. As it will be shown in Section 6, our drawing technique for 3-connected cubic planar graphs uses one of the four slopes to draw at most three edges. Thus our result positively answers the question by Dujmović et al., except for the fact that the three edges using the fourth slope are not all on the outerface and the drawing may not preserve the embedding.

Section 2 contains basic definitions and preliminaries while conclusions and open problems are discussed in Section 7.