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Strong matching preclusion for k-composition networks ☆

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ABSTRACT

Let $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$, $\cdots$, $G_k=(V_k,E_k)$ be k graphs, and let $f_1:V_1\to V_2$, $f_2:V_2\to V_3$, $\cdots$, $f_{k-1}:V_{k-1}\to V_k$, $f_k:V_k\to V_1$ be k bijections. The k-composition networks G induced by $G_1, G_2, \cdots, G_k$ is the graph with $V(G)=\bigcup_{t=1}^{k} V(G_t)$ and $E(G)=\bigcup_{t=1}^{k} E(G_t)\cup\{(a_t,f_t(a_t)):a_t\in V(G_t)$ and $1\le t\le k\}$. Many interconnection networks such as n-dimensional torus networks, recursive circulant graphs and Cayley graphs on abelian groups generated by minimal generating sets are special k-composition networks.
The strong matching preclusion number of a graph is the minimum number of edges and/or vertices whose deletion results in the remaining graph has neither perfect matchings nor almost perfect matchings. In this paper, we study the strong matching preclusion number and strong matching preclusion sets for k-composition networks with odd order. Our results generalize the main conclusion in [12].

1. Introduction

A *matching* of a graph is a set of pairwise nonadjacent edges. For a graph with n vertices, a matching M is called a *perfect matching* if its size $|M|=\frac{n}{2}$ for n is even, or an *almost perfect matching* if $|M|=\frac{n-1}{2}$ for n is odd. A graph is *matchable* if it has either a perfect matching or an almost perfect matching. Otherwise, it is called not matchable. A set F of edges in a graph G is called a *matching preclusion set* (MP set for short) if $G-F$ is not matchable. The *matching preclusion number* of G, denoted by $mp(G)$, is defined to be the minimum size of all possible such sets of G. The concept of matching preclusion was presented by Brigham et al. [2] and further studied in [6,13,15–18,21]. An obvious application of the matching preclusion problem was addressed in [2]: when each node of interconnection networks is demanded to have a special partner at any time, those that have larger matching preclusion numbers will be robust in the event of link failures.

As an extensive form of matching preclusion, the problem of strong matching preclusion was put forward by Park et al. [19] and further studied in [3–5,7–12,20,22]. A set F of vertices and/or edges in a graph G is called a *strong matching preclusion set* (SMP set for short) if $G-F$ is not matchable. The *strong matching preclusion number* of a graph G, denoted by $smp(G)$, is the minimum cardinality of all strong matching preclusion sets of G. The minimum SMP set of G is any SMP set whose size is $smp(G)$. We define $smp(G)=0$ if G is not matchable.

When a set F of vertices and/or edges is removed from a graph, the set is called a fault set. For any vertex $v\in V(G)$, let $N_G(v)$ be a set of neighboring vertices adjacent to v, and $I_G(v)$ be a set of edges incident with v. Clearly, if a graph G with an even number of vertices has an isolated vertex, then G has no perfect matchings.

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Proposition 1.1. [19] *Given a fault vertex set $X(v) \subseteq N_G(v)$ and a fault edge set $Y(v) \subseteq I_G(v)$, $X(v) \cup Y(v)$ is an SMP set of G if (a) $w \in X(v)$ if and only if $(v, w) \notin Y(v)$ for every $w \in N_G(v)$, and (b) the number of vertices in $G \backslash (X(v) \cup Y(v))$ is even.*

Any SMP set constructed as specified in Proposition 1.1 is called *trivial*. If every minimum SMP set of G is trivial, then G is called *super strong matched*. It is easy to see that, for an arbitrary vertex of degree at least one, there always exists a trivial SMP set that isolates the vertex. This observation leads to the following fact.

Proposition 1.2. [19] *For any graph G with no isolated vertices, $smp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

If $smp(G) = \delta(G)$, then G is called *maximally strong matched*.

2. Preliminaries

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $\cdots$, $G_k = (V_k, E_k)$ be k graphs, and let $f_1 : V_1 \to V_2$, $f_2 : V_2 \to V_3$, $\cdots$, $f_{k-1} : V_{k-1} \to V_k$, $f_k : V_k \to V_1$ be k bijections. The k-composition networks G induced by $G_1, G_2, \cdots, G_k$ is the graph with $V(G) = \bigcup_{t=1}^{k} V(G_t)$ and $E(G) = \bigcup_{t=1}^{k} E(G_t) \cup \{(a_t, f_t(a_t)) : a_t \in V(G_t)$ and $1 \leq t \leq k\}$. Many interconnection networks such as n-dimensional torus networks, recursive circulant graphs and Cayley graphs on abelian groups generated by minimal generating sets are special k-composition networks.

Let G be a k-composition networks induced by $G_1, G_2, \cdots, G_k$. For any vertex $a_i \in V(G_i)$, let $f_i(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$ and $f_k(a_k) = a_1^c$, $f_1(a_1^c) = a_2^c$, $f_k^{-1}(a_1) = \bar{a}_k$, $f_{k-1}^{-1}(\bar{a}_k) = \bar{a}_{k-1}$. Note that a_1^c may be not equal to a_1, and $\bar{a}_k$ may be not equal to a_k. For $1 \leq i, j \leq k$, we use $[i, j]$ to denote a set of integers: $[i, j] = \{l : i \leq l \leq j\}$ if $i < j$, and $[i, j] = \{l : i \leq l \leq k$ or $1 \leq l \leq j\}$ if $i > j$. Graph $G[i, j]$ is the subgraph of G, which is induced by $\{a_l : a_l \in V(G_l), l \in [i, j]\}$. Let $M_{i,i+1} = \{(a_i, a_{i+1}) : a_i \in V(G_i)$ and $i \in [1, k-1]\}$ and $M_{k,1}$ (or $M_{k,k+1}) = \{(a_k, a_1^c) : a_k \in V(G_k)\}$. Clearly, $M_{t,t+1}$ is a perfect matching of $G[t, t+1]$ for each $t \in [1, k]$.

Let F and F' be fault set and fault vertex set of G, respectively. Assume $F_t = F \cap \{V(G_t) \cup E(G_t)\}$, $F'_t = F' \cap V(G_t)$ and $F_{t,t+1}$ are fault edge sets in $M_{t,t+1}$ for each $t \in [1, k]$. Clearly, $F = (\bigcup_{t=1}^{k} F_t) \cup (\bigcup_{t=1}^{k} F_{t,t+1})$. The fault set in F but not in F_t is denoted by $F \backslash F_t$. The number of vertices of a graph G is its order, written $|G|$.

3. Main results

In this paper, we study the strong matching preclusion number and strong matching preclusion sets for k-composition networks with odd order, and obtain the following result:

Theorem 3.1. *Let G be a k-composition networks induced by $G_1, G_2, \cdots, G_k$, where $k(\geq 3)$ is an odd integer. Assume G_t is an r-regular connected graph such that $r(\geq 4)$ is even and $|G_t|(\geq r+5)$ is odd for each $t \in [1, k]$. If G_t is maximally strong matched for each $t \in [1, k]$, then G is super strong matched.*

For readability of the proof of Theorem 3.1, some subcases in it are solved in Lemma 3.2. For writing convenience, we will abbreviate almost perfect matching, perfect matching and perfect matchings to ap-m, p-m and p-ms, respectively. We use $M(G)$ and $M'(G)$ to represent a p-m of G and an ap-m of G, respectively.

Lemma 3.2. *Let G be a k-composition networks induced by $G_1, G_2, \cdots, G_k$, where $k(\geq 3)$ is an odd integer. Assume, for each $t \in [1, k]$, G_t is an r-regular connected graph such that $r(\geq 4)$ is even, $|G_t|(\geq r+5)$ is odd, and $smp(G_t) = r$. Let F be a fault set of G with $|F| \leq r+2$.*

• For $1 \leq i < j \leq k$, $G[i, j] - F$ is matchable if either

(1) $|F_t| < r$ and $|G_t - F_t|$ is even for each $t \in [i, j]$, or

(2) $|F_i| \leq r-1$, $|F_j| \leq r-2$ and $|F_t| \leq r-3$ for each $t \in [i+1, j-1]$. In addition, $|G_i - F_i|$ is odd and $|G_t - F_t|$ is even for each $t \in [i+1, j-1]$. Particularly, when $|F_i| = r-1$, the inequality $|F_{i,i+1} \cup F_{i+1}| \leq 1$ holds, or

(3) $|F_i| \leq r-1$, $|F_j| \leq r-2$ and $|F_t| \leq r-3$ for each $t \in [i+1, j-1]$. Particularly, when $|F_i| = r-1$ and $|G_i - F_i|$ is odd, the inequality $|F_{i,i+1} \cup F_{i+1}| \leq 1$ holds.

• For $1 \leq i < j < l \leq k$, $G[i, l] - F$ is matchable if

(4) $|F_i| \leq r-1$, $|F_j| \leq r-2$, $|F_l| \leq r-2$ and $|F_t| \leq r-3$ for each $t \in [i+1, j-1] \cup [j+1, l-1]$. Particularly, when $|F_i| = r-1$ and $|G_i - F_i|$ is odd, the inequality $|F_{i,i+1} \cup F_{i+1}| \leq 1$ holds; when $|F_j| = r-2$ and $|G[i, j] - F|$ is odd, the inequality $|F_{j,j+1} \cup F_{j+1}| \leq 1$ holds.

Proof. **(1)** Since $|F_t| < smp(G_t)$ and $|G_t - F_t|$ is even for each $t \in [i, j]$, we see that $G_t - F_t$ has a p-m M_t. So $\bigcup_{t=i}^{j} M_t$ is a p-m of $G[i, j] - F$.

(2) We first claim that $G_i - F_i$ has an ap-m M_i that misses a vertex a_i and $(a_i, a_{i+1}) \in E(G - F)$. If $|F_i| \leq r-2$, by $|M_{i,i+1}| - |F| > 1$, we can assume $(a_i, a_{i+1}) \in E(G - F)$. Since $|F_i \cup \{a_i\}| < smp(G_i)$ and $|G_i - F_i - a_i|$ is even, we see that $G_i - F_i - a_i$ has a p-m M_i. It follows that M_i is an ap-m of $G_i - F_i$ that misses vertex a_i and $(a_i, a_{i+1}) \in E(G - F)$. Consider

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