Theoretical Computer Science ••• (••••) •••–•••

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Theoretical Computer Science

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Structure fault tolerance of hypercubes and folded hypercubes ☆

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ARTICLE INFO

Article history:
Received 5 July 2017
Received in revised form 26 September 2017
Accepted 31 October 2017
Available online xxxx
Communicated by S.-Y. Hsieh

Keywords:
Structure connectivity
Substructure connectivity
Fault tolerance
Hypercube
Folded hypercube

ABSTRACT

Let G be a graph and T be a certain connected subgraph of G. The T-structure connectivity $\kappa(G;T)$ (resp. T-substructure connectivity $\kappa^s(G;T)$) of G is the minimum number of a set of subgraphs $F=\{T_1,T_2,...,T_m\}$ (resp. $F=\{T'_1,T'_2,...,T'_m\}$) such that T_i is isomorphic to T (resp. T'_i is a connected subgraph of T) for every $1\le i\le m$, and F's removal will disconnect G. Let Q_n and FQ_n denote the n-dimensional hypercube and folded hypercube, respectively. In [12], the $\kappa(Q_n;T)$ and $\kappa^s(Q_n;T)$ were determined for $T\in\{K_{1,1},K_{1,2},K_{1,3},C_4\}$. In this paper, we generalize the above results by determining $\kappa(Q_n;T)$ and $\kappa^s(Q_n;T)$ for $T\in\{P_k,C_{2k},K_{1,4}\}$ where $3\le k\le n$. We also determine $\kappa(FQ_n;T)$ and $\kappa^s(FQ_n;T)$ for $T\in\{P_k,C_{2k},K_{1,3}\}$ where $n\ge 7$ and $2\le k\le n$.

1. Introduction

Interconnection networks play an important role in parallel and distributed systems. An interconnection network can be represented by an undirected graph $G=(V,E)$, where each node in V corresponds to a processor, and every edge in E corresponds to a communication link.

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. The connectivity is one of the most important parameters to measure the reliability and fault tolerance of an interconnection network [3]. The larger the connectivity is, more reliable the interconnection network is.

However, this parameter has a deficiency. Note that, in the event of a random node failure, it is very unlikely that all of the nodes adjacent to a single node fail simultaneously. To more accurately measure the fault tolerance of an interconnection network, Harary [7] introduced the concept of conditional connectivity by attaching some conditions on connected components. Furthermore, Latifi et al. [10] generalized the concept conditional connectivity by introducing restricted h-connectivity.

Following this trend, Fábrega [6] proposed the concept of g-extra connectivity. The *g-extra connectivity* of a graph G, denoted by $\kappa_g(G)$, is the minimum number of nodes of G whose deletion disconnects G and every remaining component has more than g nodes. For the recent results on the g-extra connectivity of graphs see, for example, [2],[9],[17–20] and the references therein.

☆ The research is supported by NSFC (Nos. 11531011, 11361060) and NSFXJ (No. 2015KL019).
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https://doi.org/10.1016/j.tcs.2017.10.032
0304-3975/

So far, most researches about network reliability and fault tolerance have concentrated on the effect of individual nodes becoming faulty. In other words, it is usually assumed that the status of each individual node v, with regarded to the nodes around v, is an independent event. Nevertheless, nodes that are connected could affect one another, and the nodes closer to a faulty node are more likely to become faulty (than other distant nodes). It should also be noted that with the development of technology, networks and subnetworks are made into chips. That is to say, when any node or nodes on the chip become faulty, the whole chip could be considered faulty. All of these motivate the study of fault tolerance of networks from the perspective of some structures instead of basing on individual nodes. Under this consideration, Lin et al. [12] introduced the concept of structure connectivity and substructure connectivity of graphs.

For graph definition and notation not mentioned here we follow [1]. The *neighborhood* $N_G(v)$ of a node v in a graph $G = (V, E)$ is the set of nodes adjacent to v. For $S \subset V$, the neighborhood $N_G(S)$ of S in G is defined as $N_G(S) = (\cup_{v \in S} N_G(v)) - S$. We use $P_k = \langle v_1, v_2, \cdots, v_k \rangle$ and $C_k = \langle v_1, v_2, \cdots, v_k, v_1 \rangle$ to denote a path and a cycle of order k, respectively. For a given path P_k, we use P_k^{-1} to define the reverse of P_k, that is, $P_k^{-1} = \langle v_k, v_{k-1}, \cdots, v_1 \rangle$. For a subgraph H of a graph G, we use $G - H$ to denote the subgraph of G induced by $V(G) - V(H)$. For a set $F = \{T_1, T_2, ..., T_m\}$, where each T_i is isomorphic to a connected subgraph of G, we use $G - F$ to denote the subgraph of G induced by $V(G) - V(T_1) - V(T_2) - \cdots - V(T_m)$.

Let T be a connected subgraph of a graph G, and let F be a set of subgraphs of G such that every element in F is isomorphic to T. Then F is called a *T-structure-cut* if $G - F$ is disconnected. The *T-structure connectivity* $\kappa(G; T)$ of G is defined as the cardinality of a minimum T-structure-cut of G. Similarly, let F be a set of subgraphs of G such that every element in F is isomorphic to a connected subgraph of T. Then F is called a *T-substructure-cut* if $G - F$ is disconnected. The *T-substructure connectivity* $\kappa^s(G; T)$ of G is defined as the cardinality of a minimum T-substructure-cut of G. By definition, $\kappa^s(G; T) \le \kappa(G; T)$. Note that K_1-structure connectivity and K_1-substructure connectivity reduce to the classical node connectivity.

Lin et al. studied $\kappa(Q_n; T)$ and $\kappa^s(Q_n; T)$ for the hypercube Q_n and $T \in \{K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$ [12]. Moreover, Lv et al. investigated the Hamiltonian cycle and path embedding problems in k-ary n-cubes based on structure faults [14]. In this paper, we generalize the results in [12] and consider similar problems for the folded hypercube FQ_n.

The rest of the paper is structured as follows. In Section 2, we determine $\kappa(Q_n; T)$ and $\kappa^s(Q_n; T)$ for $T \in \{P_k, C_{2k}, K_{1,4}\}$ where $3 \le k \le n$. In Section 3, we determine $\kappa(FQ_n; T)$ and $\kappa^s(FQ_n; T)$ for $T \in \{P_k, C_{2k}, K_{1,3}\}$ where $n \ge 7$ and $2 \le k \le n$. We draw conclusion in Section 4.

2. Hypercubes

The *n-dimensional hypercube* Q_n has the node set consisting of 2^n binary strings of length n, two nodes being joined by an edge if and only if they differ in exactly one position. The hypercube Q_n possesses many attractive properties, for example, both diameter and connectivity are n, and it is bipartite and thus contains no odd cycle [8].

For any node $u = u_1 u_2 ... u_n$ in Q_n, we use u^i to denote the neighbor of u in dimension i. Similarly, $u^{i,j}$ is the neighbor of u^i in dimension j. Obviously, $u^{i,j} = u^{j,i}$.

In what follows, we will explore T-structure connectivity and T-substructure connectivity of Q_n for $T \in \{P_k, C_{2k}, K_{1,4}\}$ where $3 \le k \le n$. We first supply some lemmas for later use.

Lemma 2.1. *[17] Any two nodes in $Q_n (n \ge 3)$ have exactly two common neighbors if they have any.*

Lemma 2.2. *Let P_k be a path in Q_n with $1 \le k \le n$. If v is a node of $Q_n - P_k$, then $|N_{Q_n}(v) \cap V(P_k)| \le \lceil \frac{k}{2} \rceil$.*

Proof. Since Q_n is triangle free, v can be adjacent to at most one node of any two consecutive nodes on P_k. Thus, the lemma follows. □

Lemma 2.3. *Let P_k be a path in Q_n with $3 \le k \le n$. If u and v are two adjacent nodes of $Q_n - P_k$, then $|N_{Q_n}(\{u, v\}) \cap V(P_k)| \le k - 1$.*

Proof. Clearly, $|N_{Q_n}(\{u, v\}) \cap V(P_k)| \le k$. So it suffices to show that there exists at least one node on P_k, which is adjacent to neither u nor v. In deed, since Q_n is triangle free and by Lemma 2.1, at least one node of any three consecutive nodes on P_k is adjacent to neither u nor v. It implies that $|N_{Q_n}(\{u, v\}) \cap V(P_k)| \le k - 1$. □

Since Q_n is triangle free and by Lemma 2.1, the following lemma is straightforward.

Lemma 2.4. *Let $K_{1,r}$ be a star in Q_n with $3 \le r \le n$. If u is a node of $Q_n - K_{1,r}$, then $|N_{Q_n}(u) \cap V(K_{1,r})| \le 2$.*

Lemma 2.5. *Let $K_{1,r}$ be a star in Q_n with $3 \le r \le n$. If u and v are two adjacent nodes of $Q_n - K_{1,r}$, then $|N_{Q_n}(\{u, v\}) \cap V(K_{1,r})| \le 2$.*

Proof. Let $V(K_{1,r}) = \{\{x, x_1, x_2, \cdots, x_r\} | (x, x_i) \in E(Q_n)$ for $i \in \{1, 2, \cdots, r\}\}$. By Lemma 2.4, $|N_{Q_n}(u) \cap V(K_{1,r})| \le 2$. Without loss of generality, we may assume that $(u, x_1), (u, x_2) \in E(Q_n)$. If $(v, x) \in E(Q_n)$, then $N_{Q_n}(u) \cap N_{Q_n}(x) = \{x_1, x_2, v\}$, which contradicts Lemma 2.1. If $(v, x_j) \in E(Q_n)$ for $j \in \{3, \cdots, r\}$, then $\langle u, v, x_j, x, x_1, u \rangle$ forms an odd cycle, a contradiction. □

Please cite this article in press as: E. Sabir, J. Meng, Structure fault tolerance of hypercubes and folded hypercubes, Theoret. Comput. Sci. (2017), https://doi.org/10.1016/j.tcs.2017.10.032