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A QPTAS for the base of the number of crossing-free structures on a planar point set ☆

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A B S T R A C T

The number of triangulations of a planar n point set S is known to be c^n, where the base c lies between 2.43 and 30. Similarly, the number of crossing-free spanning trees on S is known to be d^n, where the base d lies between 6.75 and 141.07. The fastest known algorithm for counting triangulations of S runs in $2^{(1+o(1))\sqrt{n}\log n}$ time while that for counting crossing-free spanning trees runs in $O^*(7.125^n)$ time. The fastest known, non-trivial approximation algorithms for the number of triangulations of S and the number of crossing-free spanning trees of S, respectively, run in time subexponential in n. We present the first non-trivial approximation algorithms for these numbers running in quasi-polynomial time. They yield the first quasi-polynomial approximation schemes for the base of the number of triangulations of S and the base of the number of crossing-free spanning trees on S, respectively.

1. Introduction

By a crossing-free structure in the Euclidean plane, we mean a *planar straight-line graph* (PSLG), i.e., a plane graph whose edges $\{v, u\}$ are represented by properly non-intersecting straight-line segments with endpoints v, u, respectively. Triangulations and crossing-free spanning trees on finite planar point sets are the two most basic examples of crossing-free structures in the plane, i.e., PSLGs. The problems of counting the number of such structures for a given planar n-point set belong to the most intriguing in Computational Geometry [2,4,6,8,11,12].

1.1. Counting triangulations

A *triangulation* of a set S of n points in the Euclidean plane is a PSLG on S with a maximum number of edges. Let $F_t(S)$ stand for the set of all triangulations of S.

The problem of computing the number of triangulations of S, i.e., $|F_t(S)|$, is easy when S is in convex position. Simply, by a straightforward recurrence, $|F_t(S)| = C_{n-2}$, where C_k is the k-th Catalan number, in this special case. However, in the

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Table 1
Bounds on the number of different types of plane graphs according to [2].

Graph type	Lower bound	Reference	Upper bound	Reference
Triangulations	$\Omega(2.43^n)$	[12]	$O(30^n)$	[11]
Spanning cycles	1		$O(54.55^n)$	[13]
Perfect matchings	$\Omega^*(2^n)$	[5]	$O(10.05^n)$	[14]
Spanning trees	$\Omega^*(6.75^n)$	[6]	$O(141.07^n)$	[8]

general case, the problem of computing the number of triangulations of S is neither known to be $\#P$-hard nor known to admit a polynomial-time counting algorithm.

It is known that $|F_t(S)|$ lies between $\Omega(2.43^n)$ [12] and $O(30^n)$ [11]. See also Table 1. Since the so called flip graph whose nodes are triangulations of S is connected [15], all triangulations of S can be listed in exponential time by a standard traversal of this graph. When the number of the so called onion layers of the input point set is constant, the number of triangulations and other crossing-free structures can be determined in polynomial time [3]. Quite recently, Alvarez and Seidel have presented an elegant algorithm for the number of triangulations of S running in $O^*(2^n)$ time [4] which is substantially below the aforementioned lower bound on $|F_t(S)|$ (the O^* notation suppresses polynomial in n factors).

Also recently, Alvarez, Bringmann, Ray, and Seidel [2] have presented an approximation algorithm for the number of triangulations of S based on a recursive application of the planar simple cycle separator [10]. Their algorithm runs in subexponential $2^{O(\sqrt{n}\log n)}$ time and over-counts the number of triangulations by at most a subexponential $2^{O(n^{\frac{3}{4}}\sqrt{\log n})}$ factor. It also yields a subexponential-time approximation scheme for the base of the number of triangulations of S, i.e., for $|F_t(S)|^{\frac{1}{n}}$. The authors of [2] observe also that just the inequalities $\Omega(2.43^n) \le |F_t(S)| \le O(30^n)$ imply that the quantity $O(\sqrt{30 \times 2.43}^n)$ trivially computable in polynomial time approximates $|F_t(S)|$ within a large exponential factor of $O(\sqrt{30/2.43}^n)$.

Very recently, Marx and Miltzow [9] have presented an algorithm that computes the number of triangulations of S in $n^{(1+o(1))\sqrt{n}}$ time. Thus, they have significantly improved the aforementioned $O^*(2^n)$ upper bound due to Alvarez and Seidel [4]. Their algorithm relies on planar separators of size $O(\sqrt{n})$ in a triangulation similarly as that approximation one due to Alvarez et al. They identify such separators in a canonical way by a decomposition of the triangulation into nested layers. They could also extend their algorithm to include counting other crossing-free structures in $n^{(1+o(1))\sqrt{n}}$ time, e.g.: Hamilton cycles, spanning trees, perfect matchings, 3-colorable triangulations, connected structures, cycle decompositions, quadrangulations, 3-regular structures.

1.2. Counting crossing-free spanning trees

A (crossing-free) *spanning tree U* on a set S of n points in the Euclidean plane is a connected PSLG on S that is cycle-free, equivalently, that has $n-1$ edges. Let $F_s(S)$ stand for the set of all crossing-free spanning trees on S.

It is known that $|F_s(S)|$ lies between $\Omega(6.75^n)$ [6] and $O(141.07^n)$ [8]. See also Table 1. The fastest known algorithm for computing $|F_s(S)|$ runs in $O^*(7.125^n)$ time [16].

The aforementioned approximation algorithm for $|F_t(S)|$ due to Alvarez, Bringmann, Ray, and Seidel can be adapted to compute $|F_s(S)|$ approximately within the same asymptotic subexponential $2^{O(n^{\frac{3}{4}}\sqrt{\log n})}$ factor in the same asymptotic subexponential $2^{O(\sqrt{n}\log n)}$ time [2]. The adaption also yields a subexponential-time approximation scheme for the base of the number of crossing-free spanning trees on S, i.e., for $|F_s(S)|^{\frac{1}{n}}$.

1.3. Our contributions

We take a similar approximation approach to the problems of counting triangulations of S and counting crossing-free spanning trees on S as Alvarez, Bringmann, Ray, and Seidel in [2]. However, importantly, instead of using recursively the planar simple cycle separator [10], we shall apply recursively the so called balanced α-cheap l-cuts of independent sets of triangles within a dynamic programming framework developed by Adamaszek and Wiese in [1]. By using the aforementioned techniques, the authors of [1] designed the first quasi-polynomial time approximation scheme for the maximum weight independent set of polygons belonging to the input set of polygons with poly-logarithmically many edges.

Observe that a triangulation of S can be viewed as a maximum independent set of triangles drawn from the set of all triangles with vertices in S that are free from other points in S (triangles, or in general polygons, are identified with their open interiors). Also, a crossing-free spanning tree on S can be easily complemented to a full triangulation on S. These simple observations enable us to use the aforementioned balanced α-cheap l-cuts recursively in order to bound an approximation factor of our approximation algorithm. The parameter α specifies the maximum fraction of an independent set of triangles that can be destroyed by the l-cut, which is a polygon with at most l vertices in a specially constructed set of points of polynomial size.

Similarly as the approximation algorithm from [2], our algorithm may over-count the true number of triangulations or crossing-free spanning trees because the same triangulation or spanning tree, respectively, can be partitioned recursively in