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Fixing improper colorings of graphs ☆

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ABSTRACT

In this paper we consider a variation of a recoloring problem, called the Color-Fixing. Let us have some non-proper r-coloring φ of a graph G. We investigate the problem of finding a proper r-coloring of G, which is "the most similar" to φ, i.e., the number k of vertices that have to be recolored is minimum possible. We observe that the problem is NP-complete for any fixed $r \geq 3$, even for bipartite planar graphs. Moreover, it is $W[1]$-hard even for bipartite graphs, when parameterized by the number k of allowed recoloring transformations. On the other hand, the problem is fixed-parameter tractable, when parameterized by k and the number r of colors.
We provide a $2^n \cdot n^{\mathcal{O}(1)}$ algorithm for the problem and a linear algorithm for graphs with bounded treewidth. We also show several lower complexity bounds, using standard complexity assumptions. Finally, we investigate the *fixing number* of a graph G. It is the minimum k such that k recoloring transformations are sufficient to transform *any* coloring of G into a proper one.

1. Introduction

Many problems in real-life applications have a dynamic nature. When the constraints change, the previously found solution may no longer be optimal or even feasible. Therefore often there is a need to recompute the solution (preferably using the old one). This variant is called a *reoptimization* and has been studied for many combinatorial problems, e.g. TSP (see Ausiello et al. [1]), Shortest Common Superstring (see Bilò et al. [2]) or Minimum Steiner Tree (see Zych and Bilò [26]). We also refer the reader to the paper of Shachnai et al. [25], where the authors describe a general model for combinatorial reoptimization.

Another family of problems, in which we deal with transforming one solution to another, is *reconfiguration*. Here we are given two feasible solutions and want to transform one into another by a series of simple transformations in such a way that every intermediate solution is feasible (see e.g. Ito et al. [17]). When we consider a reconfiguration version of the graph coloring problem, we want to transform one proper coloring into another one in such a way that at every step we

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Table 1
The summary of parameterized complexity results for the r-Fix problem and different parameters.

$r \leq 2$	$r \geq 3$	k	$k+r$	$\text{tw}+r$
P (Proposition 4)	NP-c (Theorem 8)	$W[1]$-h (Theorem 19)	FPT (Theorem 16)	FPT (Theorem 23)

Table 2
The upper and lower complexity bounds for the problem under different parameterizations. We suppress polynomial factors. Lower bounds should be read: "there is no algorithm with this complexity, unless the ETH/SETH fails".

	n	$k+r$	$\text{tw}+r$
upper bound	2^n (Corollary 11)	$(2(r-1))^k$ (Theorem 15)	r^t (Theorem 23)
lower bound	$2^{o(n)}$ (Corollary 12)	$r^{o(k/\log k)}$ (Corollary 20)	$(r-\epsilon)^t$ (Corollary 25)

can recolor just one vertex and the coloring obtained after this change is still proper. Bonsma and Cereceda showed that deciding if a proper k-coloring φ can be transformed into another k-coloring φ' is PSPACE-complete for every $r \geq 4$ [7].

A special attention has been paid to determining if a given graph G is r-*mixing*, i.e., if for any two proper r-colorings of G you can transform one into another (maintaining a proper r-coloring at each step). Cereceda et al. [8–10] characterize graphs, which are 3-mixing, and they provide a polynomial algorithm for recognizing them. There are also some results showing that a graph G is $f(G)$-mixing, where $f(G)$ is some invariant of G. For example, Jerrum [18] showed that every graph G is $(\Delta(G)+2)$-mixing. This bound was later refined by Bonamy and Bousquet [6], who proved that every graph is $(\chi_g(G)+1)$-mixing, where $\chi_g(G)$ denotes the Grundy number of G, i.e., the highest possible number of colors used by a greedy coloring of G. Clearly $\chi_g(G) \leq \Delta(G)+1$.

Another direction of research in r-mixing graphs is the maximum number of transformations necessary to obtain one r-coloring from another one, i.e., the distance between those colorings. Bonamy and Bousquet [6] show that if $r \geq \text{tw}(G)+2$ (where $\text{tw}(G)$ denotes the *treewidth* of G), then any two r-colorings of G are at distance of at most $2(n^2+n)$, while for $r \geq \chi_g(G)+1$, any two r-colorings are at distance of at most $4 \cdot \chi_g(G) \cdot n$.

A slightly different problem has been considered by Felsner et al. [14]. They also transformed one r-coloring to another one using some local changes, but did not require the initial coloring to be proper (the final one still has to be proper). Also, a vertex could be recolored to color x if it did not have any neighbor colored with x (strictly speaking, any out-neighbor, as the authors were considering directed graphs). They showed that if G is a 2-orientation (i.e., every out-degree is equal to 2) of some maximal bipartite planar graph (i.e., a plane quadrangulation), then every proper 3-coloring of G could be reached in $\mathcal{O}(n^2)$ steps from any initial (even non-proper) 3-coloring of G. Similar results hold for 4-colorings and 3-orientations of maximal planar graphs (i.e., triangulations).

In this paper we consider a slightly different problem. We start with some (possibly non-proper) r-coloring and ask for the minimum number of transformations needed to obtain a proper r-coloring (any proper r-coloring, not a specific one). We are allowed to change colors of vertices arbitrarily, provided that we recolor just one vertex in each step. We mainly focus on the computational aspects of determining if, starting with some given r-coloring of G, we can reach a proper r-coloring in at most k steps.

The paper is organized as follows. In Section 3 we show that our problem is NP-complete for any $r \geq 3$, even if the input graph is planar and bipartite (here k is a part of the input). In Section 3.2 we provide a $2^n \cdot n^{\mathcal{O}(1)}$ algorithm for the problem and show that it is essentially optimal under the ETH. In the next two Sections we focus on the parameterized complexity (we refer the reader to [11,13] for an introduction to the parameterized complexity theory). First, we present an algorithm solving the problem in time $(2(r-1))^k \cdot n^{\mathcal{O}(1)}$, which shows that our problem is FPT, when parameterized by $k+r$ (Section 4). Then we show that the problem is $W[1]$-hard, when parameterized by k only, even if the input graph is bipartite. Moreover, we present an almost tight lower bound, excluding an algorithm with running time $f(k) \cdot r^{o(k/\log k)} \cdot n^{\mathcal{O}(1)}$ for any function f, under the ETH. Finally, we show that for any $r \geq 3$, the problem does not admit a kernel parameterized in k (unless NP $\subseteq$ coNP/poly), even in the input graph is bipartite.

In Section 4.2 we provide an algorithm solving the problem for graphs with bounded treewidth and show that is it essentially optimal, under the SETH. In Tables 1 and 2 you can find a summary of the most important results of the paper.

The last section of the paper, Section 5, is purely combinatorial. We investigate the *fixing number* of G, i.e., the maximum (over all initial colorings φ) distance from φ to a proper coloring of G. We provide some combinatorial bounds and suggest directions for future research.

2. Preliminaries

For a natural number r, by $[r]$ we denote the set $\{1, 2, .., r\}$. By an r-*coloring* of a graph G we mean any assignment of natural numbers from $[r]$ (called *colors*) to vertices of G. A coloring is *proper* if no two adjacent vertices get the same color. Note that there may be some colors that are not assigned to any vertex.