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The maximum time of 2-neighbor bootstrap percolation: Complexity results

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In 2-neighborhood bootstrap percolation on a graph G, given an initially infected set of vertices of G, an infection spreads according to the following deterministic rule: once a vertex of G is infected, it remains infected forever and in consecutive rounds healthy vertices with at least 2 already infected neighbors becomes infected. Percolation occurs if eventually every vertex is infected. The maximum time $t(G)$ is the maximum number of rounds needed to eventually infect the entire vertex set. In 2015, it was proved [8] that deciding whether $t(G) \geq k$ is polynomial time solvable for $k = 2$, but is NP-Complete for $k = 4$ and, if the problem is restricted to bipartite graphs, it is NP-Complete for $k = 7$. In this paper, we solve the open questions. We obtain an $O(mn^5)$-time algorithm to decide whether $t(G) \geq 3$. For bipartite graphs, we obtain an $O(mn^3)$-time algorithm to decide whether $t(G) \geq 3$, an $O(m^2n^9)$-time algorithm to decide whether $t(G) \geq 4$ and we prove that $t(G) \geq 5$ is NP-Complete.

1. Introduction

We consider a problem in which an infection spreads over the vertices of a connected simple graph G following a deterministic spreading rule in such a way that an infected vertex will remain infected forever. Given a set $S \subseteq V(G)$ of initially infected vertices, we build a sequence $S_{(0)}, S_{(1)}, S_{(2)}, \ldots$ in which $S_{(0)} = S$ and $S_{(i+1)}$ is obtained from $S_{(i)}$ by adding to it the vertices of G which have at least 2 neighbors in $S_{(i)}$. Given a set S of vertices and a vertex v of G, let $t(G, S, v)$ be the minimum t such that v belongs to $S_{(t)}$ (let $t(G, S, v) = \infty$ if there is no such t). We say that a set $S_{(0)}$ infects G if eventually every vertex of G becomes infected, that is, there exists t such that $S_{(t)} = V(G)$. We say that S is a hull set (or a percolating set) of G if S infects G.

If S is a percolating set of G, then we define $t(G, S)$ as the minimum t such that $S_{(t)} = V(G)$. Also, define the *percolation time of* G as $t(G) = \max\{t(G, S) : S \text{ infects } G\}$.

Bootstrap percolation was introduced by Chalupa, Leath and Reich [15] as a model for certain interacting particle systems in physics. Since then it has found applications in clustering phenomena, sandpiles [21], and many other areas of statistical physics, as well as in neural networks [1] and computer science [17].

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There are two broad classes of questions one can ask about bootstrap percolation. The first, and the most extensively studied, is what happens when the initial configuration $S_{(0)}$ is chosen randomly under some probability distribution? For example, vertices are included in $S_{(0)}$ independently with some fixed probability p. One would like to know how likely percolation is to occur, and if it does occur, how long it takes.

The answer to the first of these questions is now well understood for various graphs. The existence of thresholds first appeared in papers by Holroyd, Balogh, Bollobás, Duminil-Copin and Morris [23,4,3]. Sharp thresholds have also been proved for the hypercube (Balogh and Bollobás [2], and Balogh, Bollobás and Morris [5]). There are also very recent results due to Bollobás, Smith and Uzzell [11], about the time percolation take on the discrete torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ for a randomly chosen set $S_{(0)}$.

The second broad class of questions is the one of extremal questions. For example, what is the smallest or largest size of a percolating set with a given property? The size of the smallest percolating set in the d-dimensional grid, $[n]^d$, was studied by Pete and a summary can be found in [6]. Morris [24] and Riedl [26] studied the maximum size of minimal percolating sets on the square grid and the hypercube $\{0, 1\}^d$, respectively, answering a question posed by Bollobás. However, it was proved in [16,13] that finding the smallest percolating set is NP-complete for general graphs. Another type of question is: what is the minimum or maximum time that percolation can take, given that $S_{(0)}$ satisfies certain properties? Recently, Przykucki [25] determined the precise value of the maximum percolation time on the hypercube $2^{[n]}$ as a function of n, and Benevides and Przykucki [10,9] have similar results for the square grid, $[n]^2$, also answering a question posed by Bollobás.

Here, we consider the decision version of the percolation time problem, as stated below.

PERCOLATION TIME
Input: A graph G and an integer k.
Question: Is $t(G) \geq k$?

In 2015, Benevides, Campos, Dourado, Sampaio and Silva [8], among other results, proved that deciding whether $t(G) \geq k$ is polynomial time solvable when the parameter k is fixed in 2, but is NP-Complete when the parameter k is fixed in 4 and, when we restrict the problem to allow only bipartite graphs, is NP-Complete when the parameter k is fixed in 7, thus, leaving open the questions: $t(G) \geq k$ is polynomial time solvable when the parameter k is fixed in 3? When we restrict the problem to bipartite graphs, what is the threshold, regarding the parameter k, between polynomiality and NP-completeness? In this paper, we solve these questions.

In Section 2, we show that the Percolation Time Problem is NP-Complete when restricted to bipartite graphs and $k \geq 5$. In Section 3, we discuss the similarities between the structural characterizations presented in Sections 4, 6 and 5 and the arguments used in the proofs in these sections. In Sections 4 and 6, we show that the problem is polynomial time solvable for $k = 3$ and $k = 4$ by showing a structural characterization that can be computed in time $O(mn^3)$ and $O(m^2n^9)$ respectively. For general graphs, we show in Section 5 that the problem is polynomial time solvable for $k = 3$ by showing a structural characterization that can be computed in time $O(mn^5)$.

1.1. Related works and some notation

It is interesting to notice that infection problems appear in the literature under many different names and were studied by researches of various fields. The particular case in which $r = 2$ in r-neighborhood bootstrap percolation is also a particular case of a infection problem related to convexities in graph.

A finite *convexity space* [27] is a pair $(V, \mathcal{C})$ consisting of a finite ground set V and a set $\mathcal{C}$ of subsets of V satisfying $\emptyset$, $V \in \mathcal{C}$ and if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2 \in \mathcal{C}$. The members of $\mathcal{C}$ are called *$\mathcal{C}$-convex sets* and the *convex hull* of a set S is the minimum convex set $H(S) \in \mathcal{C}$ containing S. Also, we say that a set S is a *hull set* if the convex hull of S is V.

A convexity space $(V, \mathcal{C})$ is an *interval convexity* [12] if there is a so-called *interval function* $I : \binom{V}{2} \to 2^V$ such that a subset C of V belongs to $\mathcal{C}$ if and only if $I(\{x, y\}) \subseteq C$ for every two distinct elements x and y of C. With no risk of confusion, for any $S \subseteq V$, we also denote by $I(S)$ the union of S with $\bigcup_{x,y \in S} I(\{x, y\})$. In interval convexities, the convex hull of a set S can be computed by exhaustively applying the corresponding interval function until obtaining a convex set.

The most studied graph convexities defined by interval functions are those in which $I(\{x, y\})$ is the union of paths between x and y with some particular property. Some common examples are the P_3-convexity [19], geodetic convexity [20] and monophonic convexity [18]. We observe that the spreading rule in 2-neighbors bootstrap percolation is equivalent to $S_{(i+1)} = I(S_{(i)})$ where I is the interval function which defines the P_3-convexity: $I(S)$ contains S and every vertex belonging to some path of 3 vertices whose extreme vertices are in S. For this reason, we will use the terms percolating set and hull set, which represent the same concept, interchangeably.

In geodetic convexity, where the interval of S contains S and every vertex lying in some geodesic joining two vertices of S, it was defined the *geodetic iteration number of a graph* [14,22] which is similar to the definition of percolation time.

Given a vertex v, let $N(v)$ be the set of neighbors of v and let $N[v] = N(v) \cup \{v\}$. Also, given an integer $i \geq 0$, $N_i(v)$ denotes the set of vertices at distance i of v, $N_{\geq i}(u)$ denotes the set of vertices at distance greater or equal to i of v, and $N_{\leq i}(u)$ denotes the set of vertices at distance less or equal to i of v.