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Lowness and logical depth ☆

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ABSTRACT

Bennett's concept of logic depth [3] seeks to capture the idea that a language has a lot of useful information. Thus we would expect that neither sufficiently random nor sufficiently computationally trivial sequences are deep. A question of Moser and Stephan [11] explores the boundary of this assertion, asking if there is a low computably enumerable (Bennett) deep language. We answer this question affirmatively by constructing a superlow computably enumerable Bennett deep language.

1. Introduction

Which sets (sequences/languages) contain a lot of information? When is this information useful? The area of algorithmic information theory would suggest that a random set would have a lot of information, but a sufficiently random set would have very little *useful* information. In [3], Bennett introduced a computational method of assigning meaning to having a lot of useful information.

Bennett's intuition was that sets with a lot of useful information, *deep sets*, were those with the following property. A deep set should be one for which the more time a compressor is given, the more the compressor can compress the sequence. That is, in no computably time bounded way, can we understand the complexity of the set's initial segments.

To be more precise,

Definition 1.1 *(Bennett [3]). We say that a language L is (Bennett)-*deep *(or simply "deep" when the context is clear) if for each constant c and each computable time bound $t : \omega \to \omega$, for almost all n,*

$$K^t(L \upharpoonright n) - K(L \upharpoonright n) > c.$$

We briefly recall the definitions of prefix-free Kolmogorov complexity K and its time-bounded version K^t from the previous definition. We refer the reader to Downey and Hirschfeldt [5], Li and Vitanyi [9], or Nies [13] for background material. A Turing machine, thought of as a partial function from $2^{<\omega}$ to $2^{<\omega}$, is prefix-free if no element of its domain is a

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prefix of any other element of its domain. There is a universal prefix-free Turing machine, and we fix one such machine U. Then for all $\sigma \in 2^{<\omega}$, we define the prefix-free Kolmogorov complexity of σ to be

$$K(\sigma) = \min\{|\tau| : U(\tau) = \sigma\}.$$

For some $s \in \omega$, we define

$$K_s(\sigma) = \min\{|\tau| : U(\tau) = \sigma \text{ in at most } s \text{ many steps}\}$$

and for some $t : \omega \to \omega$, we let $K^t(\sigma) = K_{t(|\sigma|)}(\sigma)$. We use $A \upharpoonright\upharpoonright n$ to denote the initial segment of A consisting of the first $n+1$ bits, following the notation of Soare [14].

Bennett proved that as we would expect, computable languages and sufficiently random ones are shallow, that is, not deep. The notions of depth has proven quite fruitful in giving insight into intrinsic information in languages, and several further variations on the notion, mainly involving orders (in place of c) and plain complexity (in place of K) have been studied. See, for instance, [1,2,4,8–10], etc. As Moser [10] showed, *all* of these notion have a common interpretation in terms of computable time bounds and compression ratios.

The goal of our paper is to answer a question raised in Moser and Stephan [11]. In [11], those authors gave a systematic analysis of the computational power of sets (as measured by the apparatus of classical computability theory, using tools like the jump operator), against notions of logical depth.

For example, Moser and Stephan extended an earlier result of Bennett by showing that a degree $\mathbf{a}$ is high (meaning $\mathbf{a}' \geq \mathbf{0}''$) if and only if $\mathbf{a}$ contains a "strongly" deep set; one with depth ratio ϵn.

One key property of deep sets is that easy sets should not be deep. Bennett proved that computable sets (and 1-random sets) are shallow, although there can be deep computably enumerable sets like the halting problem. Moser and Stephan showed that all K*-trivial* sets are shallow, where A is K-trivial iff $(\exists c)(\forall n)(K(A \upharpoonright\upharpoonright n) \leqslant K(n+1)+c)$. K-trivial sets resemble computable sets in terms of Kolmogorov complexity. They are also low in that if A is K-trivial then $A' \equiv_T \emptyset'$. In fact, that are all *superlow* in that $A' \equiv_{tt} \emptyset'$, where this denotes truth-table equivalence (see Nies [12,13] and Downey and Hirschfeldt [5], and also Kučera and Terwijn [7] for the related concept of lowness for 1-randomness).

On the other hand it was known that, at least in terms of Kolmogorov complexity, there are deep sets quite close to being computable, at least in terms of Kolmogorov complexity. That is, Lathrop and Lutz [8] showed that there are *ultracompressible* deep sets. A is ultracompressible if and only if for all computable orders[1] g,

$$(\exists c)(\forall n)(K(A \upharpoonright\upharpoonright n) \leqslant K(n+1) + g(n+1) + c).$$

For sets in general, Moser and Stephan showed that PA degrees contain deep sets, and hence there are superlow deep sets by the Superlow Basis Theorem.

The question Moser and Stephan raise is whether such low deep sets can be computably enumerable. The thing is that enumerability has a big effect on the initial segment complexity of sets. For instance, there are superlow 1-random sets R and hence superlow sets with $(\exists c)(\forall n)(K(R \upharpoonright\upharpoonright n) + c \geqslant n)$, but if A is c.e. then $(\exists c)(\forall n)(K(A \upharpoonright\upharpoonright n) \leqslant 3 \log n + c)$. Moreover, a recurrent theme in classical computability theory is that low c.e. sets have many properties very much like computable sets (Soare [14] CH IX.3: "Low sets Resemble Recursive Sets"). So it would be reasonable to guess that all low c.e. sets are shallow. Nevertheless, we will prove the following.

Theorem 1.2. *There is a superlow c.e. Bennett deep set.*

The remainder of this paper is devoted to proving Theorem 1.2. Notation is more or less standard and generally follows Soare [14] or Downey–Hirschfeldt [5].

2. The proof

Proof. We construct a c.e. set A. To make A Bennett deep, we meet for every $i \in \omega$ the requirement

R_i : *if* φ_i *is an order function, then*

$$(\forall c)(\forall^\infty m)\ K^{\varphi_i}(A \upharpoonright\upharpoonright m) > K(A \upharpoonright\upharpoonright m) + c,$$

where $\langle \varphi_i \rangle_{i<\omega}$ is an acceptable listing of all partial computable functions. We assume that we have some approximation $\langle \varphi_{i,s} \rangle_{s<\omega}$ to each φ_i such that for all s, the domain of $\varphi_{i,s}$ is an initial segment of ω. To make A low, we meet for every $e \geqslant 1$ the requirement

$$L_e : (\exists^\infty s)(\Phi_e^A(e)[s]\downarrow) \to \Phi_e^A(e)\downarrow$$

where $\langle \Phi_e \rangle_{e<\omega}$ is an acceptable listing of all Turing functionals. We will later show that A is superlow by computably bounding the number of injuries to each L-requirement.

[1] That is, $g(n)$ is nondecreasing and is unbounded.