Theoretical Computer Science 702 (2017) 48–59

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

α-Concave hull, a generalization of convex hull

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ARTICLE INFO

Article history:
Received 3 May 2017
Received in revised form 14 August 2017
Accepted 19 August 2017
Available online 24 August 2017
Communicated by P.G. Spirakis

Keywords:
Convex hull
α-Shape
α-Concave hull
Minimum area polygon
NP-complete
Approximation algorithm

ABSTRACT

Bounding hulls such as convex hull, α-shape, χ-hull, concave hull, crust, etc. offer a wide variety of useful applications. In this paper, we explore another bounding hull, namely α-concave hull, as a generalization of convex hull. The parameter α determines the smoothness level of the constructed hull on a set of points. We show that it is NP-hard to compute α-concave hull on a set of points for any $0 < \alpha < \pi$. This leads us to a generalization of Fekete work (when $\alpha = \pi$). We also introduce $\alpha - MACP$ as an NP-hard problem similar to the problem of computing α-concave hull and present an approximation algorithm for $\alpha - MACP$. The paper ends by implementing the proposed algorithm and comparing the experimental results against those of convex hull and α-shape models.

1. Introduction

The convex hull of a set of points S in the plane, is the smallest convex polygon containing S. Many algorithms have been introduced for computing the convex hull of points [1–8]. Convex hull is used in many fields such as pattern recognition, image processing, GIS, sensor networks, path planning, etc. [9–15]. Computing the convex hull of S has an optimal $O(n \log n)$ algorithm [3].

The Minimum Area Polygon (MAP) of a set of points S in the plane, is the problem of computing the smallest simple polygon containing S. The Maximum Area Polygon ($MAXP$) of a set of points S in the plane, is the problem of computing the simple polygon with maximum area that passes through all points of S. Fekete in [16] considered MAP for the grid points and denoted this problem by GAP. He demonstrated that GAP and $MAXP$ are NP-complete [16,17]. He also presented a $\frac{1}{2}$-approximation algorithm for $MXAP$ [18] and proved that there does not exist any $(\frac{2}{3} + \varepsilon)$-approximation algorithm for this problem for $0 < \varepsilon < \frac{1}{3}$, unless $P = NP$.

Here, we consider convex and concave hulls on the set of points in the plane. We introduce a generalization of convex hull and MAP called α-concave hull such that the parameter α limits the internal angles of the constructed hull. We then show that it is NP-hard to compute α-concave hull on a set of points for any $0 < \alpha < \pi$.

Bae et al. [19] used convex hull to cover a set of points with convex sets of minimum total area or perimeter size. In [20], Jin-Seo Park and Se-Jong Oh showed that for identifying the exact area occupied by a set of points, concave hull region was more useful than convex hull. The concept of Concave Hull was first introduced as (non-convex footprint) by A. Galton and M. Duckham in [21] and then it was expanded in [22]. In [20] an algorithm was presented to compute

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http://dx.doi.org/10.1016/j.tcs.2017.08.014
0304-3975/

Concave Hull in d dimensions. Concave Hull has an effective applicability in the fields of shape reconstruction [23,24], material computation [24], GIS [25], dataset classification [26], etc.

The α-shape is another generalization of convex hull which was introduced by Edelsbrunner in [27]. The α-shape has expanded applicability in shape reconstruction [28], space decomposition [29], sensor networks [30], bioinformatics [31], feature detection [32], 3D visualization (of brain tumor) [33], etc.

χ-hull and *Crust* are another non-convex hulls to cover a set of points. The *Crust* concept was developed as a graph on the set of points in R^2 [34] and R^3 [35] and χ-hull was introduced in [36] as a simple subpolygon of Delaunay triangulation of the set of points.

The α-concave hull on a set of points in the plane is a non-convex hull with angular constraints under the minimum area condition. For $\alpha = 0$, computing α-concave hull is equivalent to that of computing convex hull with $O(n \log n)$ optimal algorithm. For $\alpha = \pi$, this problem converts to MAP as it is proved to be NP-complete.

In this paper, we put forward an NP-hard problem called Maximum Area Clustering Problem ($\alpha - MACP$) on a set of points which produces the same results as the α-concave hull. We also present a $\frac{1}{4}$-approximation algorithm for $\alpha - MACP$.

The structure of the paper is as follows. In section 2 we unveil the concept of α-polygon and α-concave hull and prove that computing an α-polygon with a given area on a set of points is an NP-complete problem. Likewise, we demonstrate that it is NP-hard to compute α-concave hull on a set of points for any $0 < \alpha < \pi$. In section 3 we introduce $\alpha - MACP$ as a problem equivalent to that of computing α-concave hull and an approximation algorithm is presented to deal with it. This is then followed by a discussion on the factor of the approximation algorithms for α-concave hull. Section 4 is devoted to experimental work and implementing our proposed approximation algorithm. We then compare the results achieved by this solution against known methods that employ convex hull and α-shape to verify its outperformance. Section 5 concludes the paper.

2. α-Concave hull

As was mentioned in the previous section, many concepts have been studied as concave hull of a set of points. Previous studies on concave hull did not consider any limitation on angles nor area of the constructed polygons. In this section we disclose the concept of α-concave hull as a concave hull with angular constraint under the minimum area condition. Let us first define the concept of α-polygon as follows:

Definition 2.1. Let A be a simple polygon. A is called α-polygon if all internal angles of A are equal or less than $\pi + \alpha$.

Definition 2.2. α-Concave hull on a set of points S is an α-polygon containing S with minimum area such that all the vertices of the polygon are a subset of S.

Based on those definitions, when $\alpha = 0$, α-polygon is equal to a convex polygon, hence, the α-concave hull of S for $\alpha = 0$ is equal to convex hull of S. In the case of $\alpha = \pi$, α-polygon is equal to a simple polygon, hence, computing α-concave hull of S for $\alpha = \pi$ converts to MAP for S. Consequently, α-concave hull definition is a generalization of the convex hull concept and computing α-concave hull, is a generalization of MAP. The computing convex hull problem has $O(n \log n)$ optimal algorithm while MAP is NP-complete.

Fig. 1 illustrates α-concave hulls on a set of points where we consider various values for the parameter α. As stated above, when $\alpha = 0$, α-concave hull is equal to convex hull of the points. For $\alpha = 12°$, concave angles would be constructed in α-concave hull. When $\alpha > 120°$, the boundary of α-concave hull would pass through all of the points. When $\alpha = 180°$, α-concave hull is a simple polygon with minimum area that passes through the points which is a solution of MAP for this set of points.

Remark 1. α-Concave hull on a set of points is not unique for a fixed value of α. Fig. 2 shows an example of two different α-concave hulls on a set of points.

Based on the Remark 1, α-concave hull on a set of points is not unique for a fixed value of α. The following theorem, however, expresses that the set of boundary points on various α-concave hulls with a fixed value of α is unique on the set of points.

Theorem 2.1. *For a set of points S and a fixed value of α, if A_1 and A_2 are two α-concave hulls on S, then boundary points of A_1 and A_2 will be equal.*

Proof. Let B_1, B_2 and CH be the boundary points of A_1, A_2 and convex hull of S, respectively. By reductio ad absurdum, suppose B_1 and B_2 are not equal. Without loss of generality, we have a point called z in which $z \notin B_2$ and $z \in B_1$. So, the polygonal chain $C_1 = ax_1x_2 \ldots z \ldots x_nb$ from a to b is on B_1 such that $a, b \in CH$ are two adjacent vertices in CH. As $a, b \in B_2$, so the polygonal chain $C_2 = ay_1y_2 \ldots y_mb$ is also placed on B_2. Fig. 3 shows the chains. Since the beginning and ending vertices of the chains C_1 and C_2 are equal and $z \in C_1$ but $z \notin C_2$, C_1 and C_2 cross each other at least once at a point c. As