Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Walks on tilings of polygons

Vesa Halava [a,b], Tero Harju [a,*]

[a] Department of Mathematics and Statistics, University of Turku, Finland
[b] Department of Computer Science, University of Liverpool, UK

ARTICLE INFO

Article history:
Received 3 November 2016
Received in revised form 6 February 2017
Accepted 16 February 2017
Available online xxxx

Keywords:
Tiling
Rectangular polygons
Zig-zag theorem
Hex game

ABSTRACT

In 1966 J.R. Isbell proved his algebraic Zig-Zag Theorem using a simple property of paths in a tiling of a plane rectangle. We prove here Isbell's lemma for more general tilings of plane rectangles.

1. Introduction

Our first version of Isbell's problem is stated for coloured rectangles that tile a given larger rectangle: let R be a plane rectangle that has a tiling consisting of rectangles $R_1, R_2, \ldots, R_n$, called *cells*, which can be of different sizes. In the tiling the cells may overlap only on their sides, but need not share full sides. We say that the tiling of R satisfies the *colour condition*, if

(1) the cells along the top edge of R are coloured white,
(2) the cells along the bottom edge of R are coloured grey, and
(3) each of the other cells is coloured either white or grey.

In Fig. 1(a) we have an example of a tiling of a rectangle satisfying the colour condition.

We show that there always exists a *good path*, i.e., a path from the left hand edge of R to the right hand edge such that the path goes along the bordering edges of differently coloured cells: at each point (excepting the vertices of the cells) on one side there is a white cell and on the other side there is a grey cell; see Fig. 1(b) for an example of a good path. Note that a good path need not be a unique solution for an instance of the problem.

Origin of the problem Instead of colours Isbell [5] used signs + and − to stamp the cells. More important difference to our approach is that the cells in [5] were assumed to be of one unit height so that they appeared levelled on horizontal lines. In the following we do not have any restriction on the form of the cells.

In his paper Isbell gave only an outline of proof of the existence of a good path, and he referred to his proof as 'hand waving', which is interesting because our proof will use handshaking! The first formal proof of the problem was given by

* Corresponding author.
E-mail addresses: vehalava@utu.fi (V. Halava), harju@utu.fi (T. Harju).

http://dx.doi.org/10.1016/j.tcs.2017.02.034
0304-3975/© 2017 Elsevier B.V. All rights reserved.

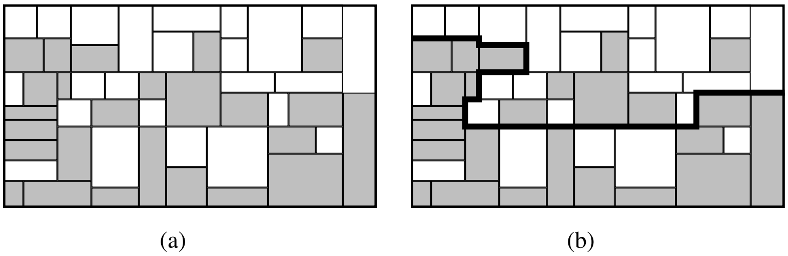

Fig. 1. In (a) a tiling of R satisfying the colour condition is shown and in (b) the bold line presents a good path.

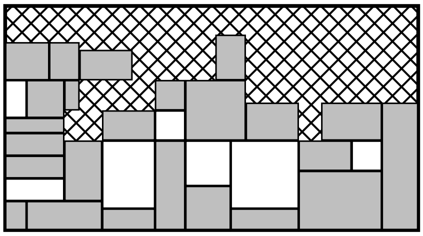

Fig. 2. The final polygon obtained by the algorithm is indicated using cross hatching.

J.M. Philip [6] in 1974. That proof employed some surplus 'inside information' of the cells inherited from the algebraic problem setting in Isbell's paper. A rather simple topological proof was given in 1997 by the present authors together with Lucian Ilie in [3]. There also an algorithm for finding a good path was given. The algorithm works also in the present more general framework of the problem, and therefore we shall shortly discuss the idea in it; for the proof, we refer to [3]. The idea is to dissolve edges of the cells as in the following:

1. merge the white cells along the top edge of R to form a polygon,
2. merge iteratively to the so obtained polygon each white cell having an edge that intersects with the edge of the polygon,
3. if there is grey polygon completely inside the unified polygon it is also added to the polygon,
4. once there are no more cells to be added to the polygon, the lowest edge line of the obtained polygon defines a desired path.

In Fig. 2 we have shown the final polygon produced by the above algorithm for the tiling in Fig. 1(a).

The problem is fairly simple to state, and the existence of a good path is almost immediate to the eye when colours are used and there are only a small number of cells. But for a mathematician it is always nice, if not a must, to find out how something fits to already established theory. In the next section we give a simple graph theoretical reasoning for the problem using a handshaking property of degrees in graphs.

2. A handshake proof

We need only the first principles of graph theory. For more advanced topics, an interested reader is referred to the book by Chartrand and Lesniak [2].

In general, a *graph* $G = (V, E)$ consists of a finite set of elements, called *vertices*, and a set E of *edges* of the form $e = \{x, y\}$, where x and y are different vertices. The *degree* $\deg(x)$ of a vertex x in G is the number of edges incident with x, i.e., $\deg(x)$ is the number of pairs $e = \{x, y\}$ in E. A vertex x is said to be *even*, if $\deg(x)$ is even, and otherwise it is called *odd*.

The following result is often referred to as the first theorem in graph theory. Its proof is imminent, since each edge provides one to the degree of each of its vertices.

Lemma 1 *(Handshaking lemma). If $G = (V, E)$ is a graph with m edges, then*

$$2m = \sum_{x \in V} \deg(x)\,.$$

In particular, every graph has an even number of odd vertices.

Let us return to our problem on rectangles. Let R be a rectangle together with a tiling by little white and grey cells as in the above. We define a graph $G_R = (V, E)$ where the set V of vertices consists of the vertices (corner points) of the cells,