Theoretical Computer Science 617 (2016) 45–64

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Unpaired many-to-many disjoint path covers in restricted hypercube-like graphs

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ARTICLE INFO

Article history:
Received 6 May 2015
Received in revised form 2 November 2015
Accepted 24 December 2015
Available online 6 January 2016
Communicated by S.-y. Hsieh

Keywords:
Hypercube-like graph
Disjoint path
Path cover
Path partition
Fault tolerance
Interconnection network

ABSTRACT

For two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of a graph, an *unpaired many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which P_i is a path from s_i to some t_j for $1 \le i, j \le k$. A family of hypercube-like interconnection networks, called *restricted hypercube-like graphs*, includes most nonbipartite hypercube-like networks found in the literature, such as twisted cubes, crossed cubes, Möbius cubes, recursive circulant $G(2^m, 4)$ of odd m, etc. In this paper, we show that every m-dimensional restricted hypercube-like graph, $m \ge 5$, with at most f faulty vertices and/or edges being removed has an unpaired many-to-many k-disjoint path cover joining arbitrary disjoint sets S and T of size k each subject to $k \ge 2$ and $f + k \le m - 1$. The bound $m - 1$ on $f + k$ is the maximum possible.

1. Introduction

An interconnection network is frequently represented as a graph in which the vertices and edges correspond to nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. One of the central issues in the study of interconnection networks is to detect (vertex-)disjoint paths, which is naturally related to routing among nodes and fault tolerance of the network [13,23]. If each copy of a message is routed along a different path of the disjoint paths, then at least one copy eventually arrives at its sink provided the total number of node and link faults is less than the number of disjoint paths. Furthermore, disjoint path is one of the fundamental notions in graph theory from which many properties of a graph can be deduced [2,23].

The connectivity of the underlying graph has been a primary measure of fault tolerance [13,23], and the connectivity of a graph is closely related to the existence of disjoint paths in the graph. Menger's theorem states the connectivity of a graph in terms of the number of disjoint paths (of one-to-one type) between a pair of source and sink, whereas the Fan Lemma states the connectivity of a graph in terms of the number of disjoint paths (of one-to-many type) joining a source to a set of sinks [2]. Moreover, a graph is k-connected if and only if it has k disjoint paths (of many-to-many type), respectively connecting arbitrary k distinct sources and arbitrary k distinct sinks, where, if a source coincides with a sink, then such source itself is regarded as a valid one-vertex path.

Let G be a simple undirected graph whose vertex and edge sets, respectively, are denoted by $V(G)$ and $E(G)$. Given two vertices v and w of G, a *path* P in G from v to w is a sequence $(u_1, \ldots, u_p)$ of distinct vertices of G such that $u_1 = v$, $u_p = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, p-1\}$. A *path cover* of G is a set of paths in G such that every vertex of G

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http://dx.doi.org/10.1016/j.tcs.2015.12.036
0304-3975/

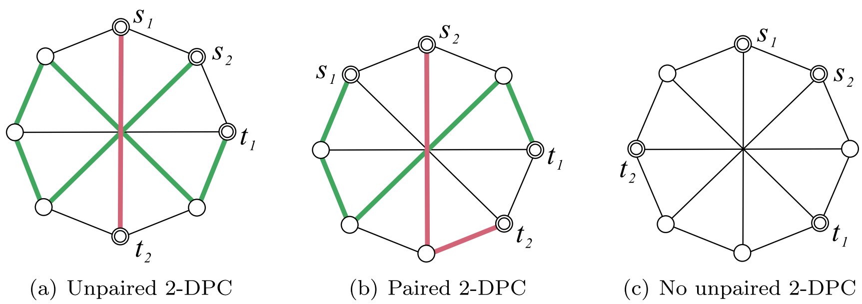

Fig. 1. Examples of unpaired and paired DPCs. The configuration (a) admits no paired 2-DPC; The configuration (c) admits no unpaired 2-DPC (and no paired 2-DPC).

is contained in at least one path. A *vertex-disjoint path cover*, or simply a *disjoint path cover*, of G is a special kind of path cover in which every vertex of G is covered by exactly one path. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1,25]. In addition, the problem is concerned with applications where full utilization of network nodes is important [32]. For example, basic communication problems for the dissemination of information, such as broadcasting and information gathering, require visiting every node of the network at least once. Since visiting a node more than once results in unnecessary overhead, a disjoint path cover can be employed to avoid this unsatisfactory situation.

For a positive integer k, let $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ be two disjoint subsets of $V(G)$. Then, a disjoint path cover $\{P_1, \dots, P_k\}$ of G is said to be an *unpaired many-to-many k-disjoint path cover* (*unpaired k-DPC* for short) joining S and T if for some permutation σ on $\{1, \dots, k\}$, P_i is a path that runs from s_i to $t_{\sigma(i)}$ for $i \in \{1, \dots, k\}$ [32]. In addition, the unpaired k-disjoint path cover is regarded as *paired* if σ is constrained to be an identity permutation, so that P_i is a path from s_i to t_i for all $i \in \{1, \dots, k\}$. Note that a paired k-DPC joining S and T is, by definition, an unpaired k-DPC joining them. Refer to Fig. 1 for examples of unpaired and paired DPCs. Here, the vertices in S and T are often called *sources* and *sinks*, respectively, and *terminals* collectively.

Definition 1. (See [33].) A graph G is called *f-fault unpaired* (resp. *paired*) *k-disjoint path coverable* if $f + 2k \le |V(G)|$ and G has an unpaired (resp. paired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \le f$.

Necessary conditions for a graph G to be f-fault unpaired k-disjoint path coverable have been derived in terms of its connectivity $\kappa(G)$ and its minimum degree $\delta(G)$ of G in [33], as shown below.

Lemma 1. *(See [33].) Let G be an f-fault unpaired k-disjoint path coverable graph, where $k \ge 2$. Then, $\kappa(G) \ge f + k$. Furthermore, if G has $f + 2k + 1$ or more vertices, then $\delta(G) \ge f + k + 1$.*

In this paper, we are concerned with the unpaired many-to-many disjoint path coverability for the class of Restricted Hypercube-Like graphs (RHL graphs for short) [31], which are a subset of nonbipartite hypercube-like graphs that have received much attention over the recent decades. The class includes most nonbipartite hypercube-like networks found in the literature, as the following examples: twisted cubes [12], crossed cubes [10], Möbius cubes [7], recursive circulant $G(2^m, 4)$ of odd m [27], multiply twisted cubes [9], Mcubes [35], and generalized twisted cubes [3]. An m-dimensional RHL graph, defined in the next section, has 2^m vertices of degree m. It is an m-regular graph of connectivity m.

Theorem 1. *(See [26].) Every m-dimensional RHL graph, $m \ge 3$, is f-fault unpaired k-disjoint path coverable for any f and $k \ge 1$ subject to $f + k \le m - 2$.*

The bound, $m - 2$, on $f + k$ in Theorem 1 is one less than the bound, $m - 1$, of the necessary condition in Lemma 1. We will bridge the gap in this paper. Precisely speaking, we will prove our main theorem asserting that *every m-dimensional RHL graph, $m \ge 5$, is f-fault unpaired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + k \le m - 1$*, achieving the optimal bound $m - 1$ on $f + k$.

The rest of this paper is organized as follows: In the next section, we address previous works and definitions. Sections 3 and 4 are devoted to a proof of our main theorem. Finally, we conclude in Section 5.