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Strong matching preclusion of (n, k)-star graphs

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ABSTRACT

The strong matching preclusion number of a graph is the minimum number of vertices and edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. This is an extension of the matching preclusion problem that was introduced by Park and Ihm. The generalized (n, k)-star graph was introduced to address scaling issues of the star graph, and it has many desirable properties. In this paper, the goal is to find the strong matching preclusion number of (n, k)-star graphs and to categorize all optimal strong matching preclusion sets of these graphs. Since bipartite graphs generally have low strong matching preclusion numbers, we assume that $k \leq n - 2$.

1. Introduction

A *perfect matching* in a graph is a set of edges such that every vertex is incident to exactly one edge in this set. An *almost-perfect matching* in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. If a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost-perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. Any such optimal set is called an *optimal matching preclusion set*. If G has neither a perfect matching nor an almost-perfect matching, then $\mathrm{mp}(G) = 0$. This concept of matching preclusion was introduced by [3] and further studied by [4–8,12,15,16,19]. This concept was introduced as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants," and extremal graph theory. We refer the readers to [3] for details. In [17], the concept of strong matching preclusion was introduced, and later studied in [2,18]. The *strong matching preclusion number* of a graph G, denoted by $\mathrm{smp}(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. Such an optimal set is called an *optimal strong matching preclusion set*.

Distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. This system topology forms the interconnection network. In particular, [11] records recent progress in this area with an extensive bibliography. In certain applications, every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures as suggested in [3]. Hence in these

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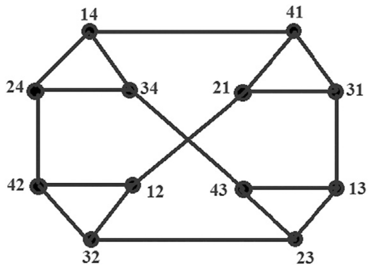

Fig. 1. $S_{4,2}$.

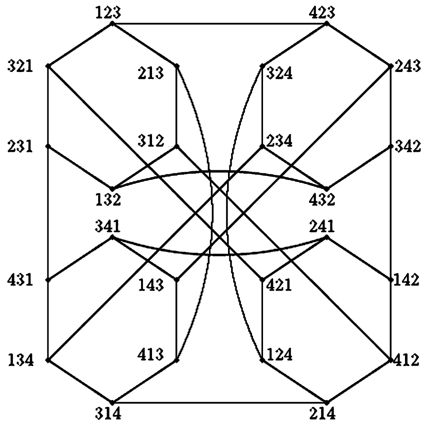

Fig. 2. $S_{4,3}$.

interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets and optimal strong matching preclusion sets are those that are "trivial." (We will define precisely what we mean by trivial later.)

The (n,k)-star graph, denoted $S_{n,k}$, is defined for positive integers n and k such that $n > k \geq 1$. The vertex set of the graph is all the permutations on k elements of the set $\{1, 2, \cdots, n\}$. Two vertices corresponding to the permutations $[a_1, a_2, \cdots, a_k]$ and $[b_1, b_2, \cdots, b_k]$ are adjacent if and only if either:

(1) There exists an integer $2 \leq s \leq k$ such that $a_1 = b_s$ and $b_1 = a_s$ and for any $i \neq s$, $1 < i \leq k$, we have $a_i = b_i$. (That is, $[b_1, b_2, \cdots, b_k]$ is obtained from $[a_1, a_2, \cdots, a_k]$ by swapping a_1 and a_s.)
(2) For all $2 \leq i \leq k$, we have $a_i = b_i$ and $a_1 \neq b_1$. (That is, $[b_1, b_2, \cdots, b_k]$ is obtained from $[a_1, a_2, \cdots, a_k]$ by replacing a_1 by an element in $\{1, 2, \ldots, n\} - \{a_1, a_2, \ldots, a_k\}$.)

See Figs. 1 and 2 for $S_{4,2}$ and $S_{4,3}$, respectively. Note that $S_{n,n-1}$ is isomorphic to the n-dimensional star graph S_n (as defined in [1]) and $S_{n,1}$ is isomorphic to the complete graph on n vertices K_n.

Let the set of vertices representing permutations whose kth element is i be called the *substar* H_i for $1 \leq i \leq n$. It can easily be seen that $S_{n,k}$ is $(n-1)$-regular as each vertex has $k-1$ neighbors by adjacency rule (1) and $n-k$ neighbors by adjacency rule (2); from another perspective, given any vertex $[a_1, a_2, \cdots, a_k]$ there are exactly $n-1$ indices distinct from a_1, and therefore $n-1$ distinct neighbors. Let us first note some other preliminary facts about $S_{n,k}$.

(1) H_i is isomorphic to $S_{n-1,k-1}$ when $n > k > 1$. This is clear from noting that removing i from all the permutations in H_i results in permutations of $k-1$ elements from $\{1, 2, \cdots, n\} - \{i\}$. This fact is highly useful in the inductive proofs of the paper, as we can often use the induction hypothesis on the H_i's.

(2) It follows from the definition that $S_{n,k}$ has $\frac{n!}{(n-k)!}$ vertices. Hence H_i has $\frac{(n-1)!}{(n-k)!}$ vertices for all $1 \leq i \leq n$.

(3) Each vertex in H_i has a unique neighbor outside of H_i called its *outside neighbor*. Consider some vertex representing the permutation $[a_1, a_2, \cdots a_{k-1}, i]$. Any neighbor outside H_i must have a different last element, and thus must apply adjacency rule (1) with $s = k$, giving the unique neighbor as $[a_k, a_2, a_3, \cdots, a_{k-1}, a_1]$. It follows that there are exactly $\frac{(n-1)!}{(n-k)!}$ cross edges between H_i and $S_{n,k} - V(H_i)$ (henceforth referred to as *cross edges* from H_1), all of which are independent. (A set of edges are *independent* if no two are incident to a common vertex.)

(4) For each pair H_i and H_j, there are exactly $\frac{(n-2)!}{(n-k)!}$ independent edges between them. Note that all edges between H_i and H_j have to result from adjacency rule (1) where permutations of the form $[j, a_2, a_3, \cdots, a_{k-1}, i]$ are adjacent to permutations of the form $[i, a_2, a_3, \cdots, a_{k-1}, j]$. Thus, the number of such pairs is the number of ways to choose a_2 through a_{k-1}, which is the number of $k-2$ element permutations on the set $\{1, 2, \cdots n\} - \{i, j\}$, or $\frac{(n-2)!}{(n-k)!}$. Clearly, these edges are independent.

(5) Seeing as each vertex has a unique outside neighbor and hence a unique cross edge, it follows that the set of cross edges forms a perfect matching on $S_{n,k}$.

2. Preliminaries

In this section, we will give some fundamental results regarding strong matching preclusion and its relationship to matching preclusion. We start with the following known general result.

Proposition 2.1. *Let G be a graph with an even number of vertices. Then $smp(G) \leq mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*