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All cartesian closed categories of quasicontinuous domains consist of domains

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ABSTRACT

Quasicontinuity is a generalisation of Scott's notion of continuous domain, introduced in the early 80s by Gierz, Lawson and Stralka. In this paper we ask which cartesian closed full subcategories exist in **qCONT**, the category of all quasicontinuous domains and Scott-continuous functions. The surprising, and perhaps disappointing, answer turns out to be that all such subcategories consist entirely of continuous domains. In other words, there are no new cartesian closed full subcategories in **qCONT** beyond those already known to exist in **CONT**.

To prove this, we reduce the notion of meet-continuity for dcpos to one which only involves well-ordered chains. This allows us to characterise meet-continuity by "forbidden substructures". We then show that each forbidden substructure has a non-quasicontinuous function space.

1. Introduction

Domain theory was introduced by Dana Scott in the late sixties as a mathematical universe within which to define the semantics of programming languages. In this programme, one seeks to identify the precise properties of domains that correspond to the language features of interest. Early on it became clear that the ability to define higher-order functions in programming has its counterpart in the formation of function spaces, more precisely, in the requirement that a category of domains be *cartesian closed*. In 1980, [17] Nasser Saheb-Djahromi considered programs with a probabilistic choice operator and in order to accommodate this in the semantics, introduced the probabilistic powerdomain construction. This was studied more deeply by Claire Jones and Gordon Plotkin, [9,8], towards the end of that decade and it was shown that the behaviour of the probabilistic powerdomain construction is much more easily understood in the context of *continuous domains*. This confirmed the experience gained before, namely, that continuous domains allow one to say much more about the workings of domain constructions and their relationship with program constructs. In a nutshell, one can say that the nice properties of continuous domains result from the fact that they are *completions of finitary structures*.

However, while it is advantageous to have continuous domains as *inputs* to domain constructions, it is not always guaranteed that one obtains them in the *output*, too. The most prominent construction which causes a complication is the function space: given two continuous domains D and E, the space $[D \to E]$ of Scott-continuous functions may not itself be continuous. For a detailed discussion of this phenomenon, see [10]. One way to overcome this problem is to restrict continuous

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domains even further and this indeed leads to various cartesian closed categories, such as the continuous Scott domains, RB-domains [12] and FS-domains [11]. However, it remains an *open problem* to find such a category that is simultaneously closed under the probabilistic powerdomain construction.

In light of these difficulties, it is natural to ask whether the condition of continuity can be relaxed, which would allow us to cast our net wider. A natural candidate for a more liberal notion of approximation is that of *quasicontinuity*, a concept which was introduced in the early eighties by Gerhard Gierz, Jimmie Lawson and Albert Stralka, [3], as a generalisation of classical continuous domains. Indeed, Jean Goubault-Larrecq, [4], was able to show that the category of *QRB-domains* (a special class of quasicontinuous domains) is closed under the probabilistic powerdomain construction, adding to what is a very small set of such closure results. This led many researchers to re-examine quasicontinuous domains [4–6,14,15,19] and many pleasing properties were established. For example, it was proved by Goubault-Larrecq and the second author, [5], and independently by Jimmie Lawson and Xiaoyong Xi, [14], that QFS-domains and QRB-domains are the same class and that they can be characterised as being precisely the Lawson-compact quasicontinuous domains, while, in the classical case, whether FS-domains and RB-domains are the same is one of the oldest and best-known open problems in domain theory.

However, unlike FS-domains or RB-domains, the category of QFS-domains (equivalently, QRB-domains) with Scott-continuous functions as morphisms is not cartesian closed (see Remark 3.9). This raises the question whether there are any new cartesian closed categories consisting of quasicontinuous domains at all. In this paper we show that this is not the case.

A note on provenance The origin of the ideas for the present paper lies in the paper [19] published in Chinese by Haoran Zhao and Hui Kou. During 2014, the same two authors went on to show that a cartesian closed category of *countably based* quasicontinuous domains must consist of continuous domains entirely. Shortly afterwards and independently, the other three authors of the present paper also considered [19] and came up with the same result about ccc's of countably based quasicontinuous domains. They continued their investigation with the aim of removing the countability assumption and as we report below this was eventually successful. The parallels between the work of the two teams were discovered during the reviewing process of our respective journal submissions and we are grateful to our editor, Michael Mislove, for encouraging us to combine our efforts into a single paper.

2. Preliminaries

We use the standard definitions of domain theory as can be found in [1] or [2]. The following, taken from [2, Section III-3], may be less familiar: One says that a subset G of a dcpo is *way-below* a subset H if for every directed set D, $\sup D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. This generalises the usual way-below relation between elements which justifies writing $G \ll H$ for it. If H consists of a single element x then one writes $G \ll x$ instead of $G \ll \{x\}$. Consistent with this we define an order between subsets G, H by $G \leq H \Leftrightarrow \uparrow H \subseteq \uparrow G$. This implies that a family $\mathcal{F}$ of subsets is *directed* if the corresponding family $\{\uparrow G \mid G \in \mathcal{F}\}$ is a filter base.

Definition 2.1. A dcpo L is called *quasicontinuous* (or a *quasicontinuous domain*) if for each $x \in L$ the family

$$\mathrm{fin}(x) = \{F \mid F \text{ is finite}, F \ll x\}$$

is a directed family, and whenever $x \not\leq y$, then there exists $F \in \mathrm{fin}(x)$ with $y \notin \uparrow F$, i.e., $\uparrow x = \bigcap\{\uparrow F \mid F \in \mathrm{fin}(x)\}$.

The following key fact relies on the Axiom of Choice via Rudin's Lemma:

Proposition 2.2. *(See [2, Proposition III-3.4].) Let $\mathcal{F}$ be a directed family of non-empty finite sets in a dcpo. If $G \ll H$ and $\bigcap\{\uparrow F \mid F \in \mathcal{F}\} \subseteq \uparrow H$, then $F \subseteq \uparrow G$ for some $F \in \mathcal{F}$.*

We use this to prove the following convenient criterion for quasicontinuity:

Proposition 2.3. *A dcpo L is quasicontinuous, if for every $x \in L$ the family $\mathrm{fin}(x)$ contains a directed subfamily $\mathcal{G}$ such that $\uparrow x = \bigcap\{\uparrow G \mid G \in \mathcal{G}\}$.*

Proof. We only need to prove that the family $\mathrm{fin}(x)$ is directed. For $F, H \in \mathrm{fin}(x)$, since $F, H \ll x$ and $\bigcap\{\uparrow G \mid G \in \mathcal{G}\} = \uparrow x$, by Proposition 2.2, there exist $G_1, G_2 \in \mathcal{G}$ such that $G_1 \subseteq \uparrow F$ and $G_2 \subseteq \uparrow H$. Then some $G \in \mathcal{G}$ is included in $\uparrow F \cap \uparrow H$ since $\mathcal{G}$ is directed. □

Proposition 2.4. *(See [2, Proposition III-3.6].) Let P be a quasicontinuous domain. A subset U of P is Scott open iff for each $x \in U$ there exists a finite $F \ll x$ such that $\uparrow F \subseteq U$. The sets $\Uparrow F = \{x \mid F \ll x\}$ are Scott open and they form a basis for the Scott topology.*

Quasicontinuity is preserved by Scott-continuous retractions: