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Real or natural number interpretation and their effect on complexity ☆

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ABSTRACT

Interpretation methods have been introduced in the 70s by Lankford [1] in rewriting theory to prove termination. Actually, as shown by Bonfante et al. [2], an interpretation of a program induces a bound on its complexity. However, Lankford's original analysis depends deeply on the Archimedean property of natural numbers. This goes against the fact that finding a real interpretation can be solved by Tarski's decision procedure over the reals (as described by Dershowitz in [3]), and consequently interpretations are usually chosen over the reals rather than over the integers. Doing so, one cannot use anymore the (good) properties of the natural (well-)ordering of **N** used to bound the complexity of programs. We prove that one may take benefit from the best of both worlds: the complexity analysis still holds even with real numbers. The reason lies in a deep algebraic property of polynomials over the reals. We illustrate this by two characterizations, one of polynomial time and one of polynomial space.

In rewriting theory, programs are described by *rules*, that is by some local conditions on the configurations and by local transformations (relative to the local conditions). Doing so, rule descriptions are usually rather elegant and very short, thus allowing some simple analyses. To comprehend the global behavior of computations—as described by a set of rewrite rules—there are two issues. First, one should extend local analyses to contexts. Second, due to the intrinsic non-determinism of rule applications imputable to locality, one configuration may lead to several computations. From a complexity point of view, some of these are 'bad', some 'good'. We shall see that replacing interpretations over natural numbers by interpretations over real numbers affects both issues.

Over properties of programs, termination is certainly one of the firsts. To prove the termination of a program, one actually proves that some measure on some well-founded order decreases along the computations. Interpretation methods were introduced to fulfill such requirements. They take benefit from the inductive structure of terms. An interpretation maps each symbol f of the rewriting system to a function $(\!| f |\!)$ in some interpretation domain. In order to get termination, Lankford describes interpretations as monotone algebras with domain of interpretation being the natural numbers with their usual ordering (cf. [1,4]).

Surprisingly, to get termination, the well-foundedness of the domain of interpretations is not mandatory as long as interpretation functions are monotonic and have the sub-term property, that is $(\!| f |\!)(x_1, \dots, x_n) \geq x_i$ for all $i \leq n$. Based on

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Kruskal's Theorem, this was shown by Dershowitz in a seminal paper [3]. Thus, in particular, the domain of the algebra mentioned above can be the set of real numbers.

There are good reasons to choose real numbers rather than natural numbers. First, we get a procedure to verify the validity of an interpretation of a program by Tarski's decomposition procedure [5] and, according to the same procedure, an algorithm to compute interpretations up to some fixed degree. Moreover, following Roy et al. [6], the complexity of these algorithms is exponential with respect to the size of the program. Actually, in most cases, one may use a faster procedure to prove the inequalities. Let us mention here the (sufficient) criterion of Hong and Jakuš [7].

A second good point is that the use of reals (as opposed to integers) provides different sets of rewriting systems that are compatible with an interpretation. As justified recently by Lucas [8], some systems are compatible with a real number interpretation but not with a natural number one. However, the contrary may happen: real interpretations do not subsume natural number ones as shown by Neurauter and Middeldorp [9].

Using natural numbers interpretations, Hofbauer and Lautemann, see [10], provided a double exponential upper bound on the length of computations of rewriting systems with a termination proof by a polynomial interpretation method. Their analysis has been deepened by Cichon and Lescanne in [11] where they stratify derivation length with respect to the shape of polynomials used for constructor symbols.

These upper bounds rely on the following argument. Given some derivation $t_1 \to t_2 \to \cdots$, since at each step the interpretation decreases, the number of steps is necessarily bounded by the interpretation of t_1. Bonfante, Cichon, Marion and Touzet [2] then showed that the above mentioned bounds could be turned into complexity characterizations. This initial contribution opened a new line of research: in the last years, the study of termination methods has been one of the major tools in implicit computational complexity. Among such results, Avanzini, Eguchi and Moser have characterized Ptime and sub-hierarchies by means of pop* in [12,13], and context dependent interpretations in [14] after their introduction by Hofbauer [15]. One of our two characterizations, namely Theorem 37, rely on the notion of dependency pairs (cf. [16]). In this vein, we mention here the work of Hirokawa and Moser [17], and, in the same spirit, Lucas and Peña in [18] made some investigations on the complexity of first order functional programs.

So, as seen above, researchers tackled more and more complex termination proof methods, more and more elaborate program paradigms. Here, we come back to the analysis of the interpretation method, but with a different view. Our thesis is that for complexity analysis of programs, due to Positivstellensatz [19]—a deep result in algebraic geometry—, polynomial interpretations over the reals can safely replace polynomial interpretations over the integers. We illustrate our claim by two theorems, Theorem 29 and Theorem 37. The two theorems are based on some preliminary results shown in Section 2, themselves being direct consequences of Positivstellensatz.

But, let us draw briefly the roadmap of the key technical features of this work. Given a strict interpretation for a term rewriting system, it follows immediately that for all rewriting steps $s \to t$, we have

$$(\!| s |\!) > (\!| t |\!). \tag{1}$$

If one takes an interpretation on natural numbers (as they were introduced by Lankford [1]), this inequality leads to a bound on the derivation height:

$$\mathtt{dh}(t) \leq (\!| t |\!). \tag{2}$$

Indeed, suppose $t_0 \to t_1 \to \cdots \to t_n$, since $(\!| t_0 |\!) > (\!| t_1 |\!) > \cdots > (\!| t_n |\!)$ are inequalities on natural numbers, this means that $n \leq (\!| t_0 |\!)$.

Hence the double-exponential bound on the derivation as given by Hofbauer and Lautemann in [10]. However (and obviously), their argument does not hold with real numbers. Can we recover it? Let us look more closely to Eq. (1). The inequality is a consequence of a) for all rewrite rule $\ell \to r$:

$$(\!| \ell |\!) \geq (\!| r |\!) + 1 \tag{3}$$

and b) for all $x_i > y_i$:

$$(\!| f |\!)(x_1, \ldots, x_i, \ldots, x_n) - (\!| f |\!)(x_1, \ldots, y_i, \ldots, x_n) \geq x_i - y_i. \tag{4}$$

The two equations (3) and (4) do not hold in general for real interpretations. To recover the Noetherian property coming with natural numbers, some authors modified the notion of order on real numbers. For instance, Lucas in [8], Marion and Péchoux in [20] or Neurauter and Middeldorp in [9] suppose the existence of some real $\delta > 0$ such that for any rule $\ell \to r : (\!| \ell |\!) \geq (\!| r |\!) + \delta$. In other words, they introduce a slightly generalized form of Eq. (3). Actually, this is a technical trick which we prove not necessary.

Due to equality (4), for all terms $t \to u$ and all contexts C, we can state $(\!| C[t] |\!) - (\!| C[u] |\!) \geq (\!| t |\!) - (\!| u |\!)$. In other words, it provides closure by context, a crucial argument to reason about derivations. Again, we will have to recover one form of this inequality.

To sum up, Dershowitz showed in [3] that termination was preserved by the shift from interpretations over the integers to interpretations over the reals. We show that complexity is also preserved if the functions used for interpretations are polynomials (with a possible extension to the max function). More generally, we think that this study provides evidence that the algebraic structure of interpreted functions plays a key role in the analysis of termination proofs by monotone interpretations.