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# On minimum average stretch spanning trees in polygonal 2-trees ☆,☆☆

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ABSTRACT

A spanning tree of an unweighted graph is a *minimum average stretch spanning tree* if it minimizes the ratio of sum of the distances in the tree between the end vertices of the graph edges and the number of graph edges. For a polygonal 2-tree on $n$ vertices, we present an algorithm to compute a minimum average stretch spanning tree in $O(n \log n)$ time. This algorithm also finds a minimum fundamental cycle basis in polygonal 2-trees. We show that there is a unique minimum cycle basis in a polygonal 2-tree and it can be computed in linear time.



## 1. Introduction

Average stretch is a parameter used to measure the quality of a spanning tree in terms of distance preservation, and finding a spanning tree with minimum average stretch is a classical problem in network design. Let $G = (V(G), E(G))$ be an unweighted graph and $T$ be a spanning tree of $G$. For an edge $(u, v) \in E(G)$, $d_T(u, v)$ denotes the distance between $u$ and $v$ in $T$. The average stretch of $T$ is defined as

$$\text{AvgStr}(T) = \frac{1}{|E(G)|} \sum_{(u,v)\in E(G)} d_T(u, v) \tag{1}$$

A *minimum average stretch spanning tree* of $G$ is a spanning tree that minimizes the average stretch. Given an unweighted graph $G$, the minimum average stretch spanning tree (Mast) problem is to find a minimum average stretch spanning tree of $G$. Due to the unified notation for tree spanners, the Mast problem is equivalent to the problem, Mfcb, of finding a minimum fundamental cycle basis in unweighted graphs [17]. Minimum average stretch spanning trees are used to solve symmetric diagonally dominant linear systems [17]. Further, minimum fundamental cycle bases have various applications including determining the isomorphism of graphs, frequency analysis of computer programs, and generation of minimal perfect hash functions (see [4,11] and the references therein]). Due to these vast applications, finding a minimum average stretch spanning tree is useful in theory and practice. The Mast problem was studied in a graph theoretic game in the context of the $k$-server problem by Alon et al. [1]. The Mfcb problem was introduced by Hubika and Syslo in 1975 [12]. The Mfcb problem was proved to be NP-hard by Deo et al. [4] and APX-hard by Galbiati et al. [11]. Another closely related problem is the problem of probabilistically embedding a graph into its spanning trees. A graph $G$ is said to be *probabilistically*

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*embedded* into its spanning trees with *distortion* $t$, if there is a probability distribution $D$ of spanning trees of $G$, such that for any two vertices the expected stretch of the spanning trees in $D$ is at most $t$. The problem of probabilistically embedding a graph into its spanning trees with low distortion has interesting connections with low average stretch spanning trees.

In the literature, spanning trees with low average stretch have received significant attention in special graph classes such as $k$-outerplanar graphs and series-parallel graphs. In case of planar graphs, Kavitha et al. remarked that the complexity of MFCB is unknown and there is no $O(\log n)$ approximation algorithm [13]. For $k$-outerplanar graphs, the technique of peeling-an-onion decomposition is employed to obtain a spanning tree whose average stretch is at most $c^k$, where $c$ is a constant [7]. In case of series-parallel graphs, a spanning tree with average stretch at most $O(\log n)$ can be obtained in polynomial time (see Section 5 in [8]). The bounds on the size of a minimum fundamental cycle basis are studied in graph classes such as planar, outerplanar and grid graphs [13]. The study of probabilistic embeddings of graphs is discussed in [7,8]. To the best of our knowledge, there is no published work to compute a minimum average stretch spanning tree and minimum fundamental cycle basis in any subclass of planar graphs.

We consider polygonal 2-trees in this work, which are also referred to as polygonal-trees. They have a rich structure that make them very natural models for biochemical compounds, and provide an appealing framework for solving associated enumeration problems.

**Definition 1.** (See [14].) A cycle is a polygonal 2-tree. For a polygonal 2-tree $G$ such that $(u,v) \in E(G)$, adding a path $P$ between $u$ and $v$ in such a way that $E(G) \cap E(P) = \emptyset$, $V(G) \cap V(P) = \{u,v\}$, and $|E(P)| \geq 2$ results in a polygonal 2-tree.

A cycle consisting of $k$ edges is a *k-gonal tree*. For a $k$-gonal 2-tree $G$ such that $(u,v) \in E(G)$, adding a path $P$ between $u$ and $v$ in such a way that $E(G) \cap E(P) = \emptyset$, $V(G) \cap V(P) = \{u,v\}$, and $|E(P)| = k-1$ results in a *k-gonal 2-tree*. For example, a *2-tree* is a 3-gonal tree. The class of polygonal 2-trees is a subclass of planar graphs and it includes 2-connected outerplanar graphs and $k$-gonal trees. 2-trees, in other words 3-gonal trees, are extensively studied in the literature. In particular, previous work on various flavors of counting and enumeration problems on 2-trees is compiled in [10]. Formulas for the number of labeled and unlabeled $k$-gonal trees with $r$ polygons (induced cycles) are computed in [15]. The family of $k$-gonal trees with same number of vertices is claimed as a chromatic equivalence class by Chao and Li, and the claim has been proved by Wakelin and Woodal [14]. The class of polygonal 2-trees is shown to be a chromatic equivalence class by Xu [14]. Further, various subclasses of generalized polygonal 2-trees have been considered, and it has been shown that they also form a chromatic equivalence class [14,19,20]. The enumeration of outerplanar $k$-gonal trees is studied by Harary, Palmer and Read to solve a variant of the cell growth problem [6]. Molecular expansion of the species of outerplanar $k$-gonal trees is shown in [6]. Also outerplanar $k$-gonal trees are of interest in combinatorial chemistry, as the structure of chemical compounds like catacondensed benzenoid hydrocarbons forms an outerplanar $k$-gonal tree.

**Our results.** We state our main theorem.

**Theorem 2.** *Given a polygonal 2-tree G on n vertices, a minimum average stretch spanning tree of G can be obtained in* $O(n \log n)$ *time.*

Due to the equivalence of MAST and MFCB (shown in Lemma 5), our result implies the following corollary. For a set $\mathcal{B}$ of cycles in $G$, the size of $\mathcal{B}$, denoted by size($\mathcal{B}$), is the number of edges in $\mathcal{B}$ counted according to their multiplicity.

**Corollary 3.** *Given a polygonal 2-tree G on n vertices, a minimum fundamental cycle basis* $\mathcal{B}$ *of G can be obtained in* $O(n \log n + \text{size}(\mathcal{B}))$ *time.*

We characterize polygonal 2-trees using a kind of ear decomposition and present the structural properties of polygonal 2-trees that are useful in finding a minimum average stretch spanning tree (in Section 3). We then identify a set of edges in a polygonal 2-tree, called safe edges, whose removal results in a minimum average stretch spanning tree (in Section 4). We present an algorithm with necessary data-structures to identify the safe set of edges efficiently and compute a minimum average stretch spanning tree in sub-quadratic time (in Section 5). We finally characterize polygonal 2-trees using cycle bases, which is of our independent interest (in Section 6).

A graph $G$ can be probabilistically embedded into its spanning trees with distortion $t$ if and only if the multigraph obtained from $G$ by replicating its edges has a spanning tree with average stretch at most $t$ (see [1]). It is easy to observe that, a spanning tree $T$ of $G$ is a minimum average stretch spanning tree for $G$ if and only if $T$ is a minimum average stretch spanning tree for a multigraph of $G$. As a consequence of our result, we have the following corollary.

**Corollary 4.** *For a polygonal 2-tree G on n vertices, the minimum possible distortion of probabilistically embedding G into its spanning trees can be obtained in* $O(n \log n)$ *time.*