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Advancements on SEFE and Partitioned Book Embedding problems

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ABSTRACT

In this work we investigate the complexity of some combinatorial problems related to the *Simultaneous Embedding with Fixed Edges* (SEFE) and the Partitioned T-Coherent k-Page Book Embedding (PTBE-k) problems, which are known to be equivalent under certain conditions. Given k planar graphs on the same set of n vertices, the SEFE problem asks to find a drawing of each graph on the same set of n points in such a way that each edge that is common to more than one graph is represented by the same curve in the drawings of all such graphs. Given a tree T with n leaves and a collection of k edge-sets E_i connecting pairs of leaves of T, the PTBE-k problem asks to find an ordering $\mathcal{O}$ of the leaves of T that is represented by T such that the endvertices of two edges in any set E_i do not alternate in $\mathcal{O}$.

The SEFE problem is $\mathcal{NP}$-complete for $k \geq 3$ even if the intersection graph is the same for each pair of graphs (*sunflower intersection*). We prove that this is true even when the intersection graph is a tree and all the input graphs are biconnected. This result implies the $\mathcal{NP}$-completeness of PTBE-k for $k \geq 3$. However, we prove stronger results on this problem, namely that PTBE-k remains $\mathcal{NP}$-complete for $k \geq 3$ even if (i) two of the input graphs $G_i = T \cup E_i$ are biconnected and T is a caterpillar or if (ii) T is a star. This latter setting is also known in the literature as Partitioned k-Page Book Embedding. On the positive side, we provide a linear-time algorithm for PTBE-k when all but one of the edge-sets induce connected graphs.

Finally, we prove that the problem of maximizing the number of edges that are drawn the same in a SEFE of two graphs (*optimization of SEFE*) is $\mathcal{NP}$-complete, even in several restricted settings.

1. Introduction

Let $G_1, \dots, G_k$ be k graphs on the same set V of vertices. A *simultaneous embedding with fixed edges* (SEFE) of $G_1, \dots, G_k$ consists of k planar drawings $\Gamma_1, \dots, \Gamma_k$ of $G_1, \dots, G_k$, respectively, such that each vertex $v \in V$ is mapped to the same point in every drawing Γ_i and each edge that is common to more than one graph is represented by the same simple curve in the drawings of all such graphs. The *SEFE problem* is the problem of testing whether k input graphs $G_1, \dots, G_k$ admit a SEFE [1].

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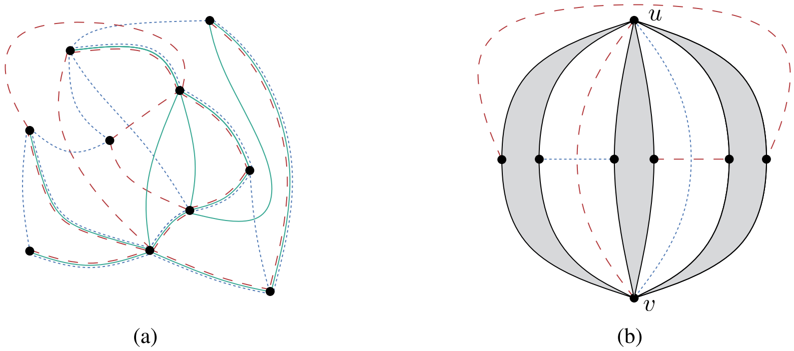

Fig. 1. (a) A Sunflower SEFE of three planar graphs. (b) A Max SEFE of two graphs. Note that edge (u, v) is represented as a different curve in the two drawings.

The possibility of drawing together a set of graphs gives the opportunity to represent at the same time a set of different binary relationships among the same objects, hence making this topic a fundamental tool in Information Visualization [2]. Motivated by such applications and by their theoretical appealing, simultaneous graph embeddings received wide research attention in the last few years. For an up-to-date survey, see [3].

Recently, a new major milestone to assert the importance of SEFE has been provided by Schaefer [4], who discussed its relationships with some other famous problems in Graph Drawing, proving that SEFE generalizes several of them. In particular, he showed a polynomial-time reduction to SEFE with $k = 2$ from the *clustered planarity testing* problem [5,6], whose computational complexity is still one of the most important open questions in Graph Drawing. Recently, the reduction in the opposite direction has been proved [7], but only for instances of SEFE of two graphs in which the *intersection graph*, namely the graph composed of edges that belong to both graphs, is connected. We remark that this "connected" version of SEFE is equivalent to problem Partitioned T-Coherent k-Page Book Embedding (PTBE-k) for $k = 2$, that is defined [8] as follows. Given a set V of vertices, a tree T whose leaves are the elements of V, and a collection of edge-sets $E_i \subseteq V \times V$, for $i = 1, \dots, k$, is there an ordering $\mathcal{O}$ of the elements of V such that (i) the ordering $\mathcal{O}$ is represented by T and (ii) the endvertices of any two edges belonging to the same set E_i do not alternate in $\mathcal{O}$? Intuitively, the problem aims at placing the vertices along the spine of a book in such a way that (i) the placement is consistent with tree T and (ii) it is possible to draw the edges of each set on a page of the book without creating any crossing.

For $k = 2$, the complexity of SEFE is still unknown, even in its "connected" version PTBE-2; however, polynomial-time algorithms exist for instances $\langle G_1, G_2 \rangle$ in which: (i) one of G_1 and G_2 has a fixed embedding [9]; (ii) the intersection graph $G_\cap$ is composed of one [8,10] or more [11] biconnected components, a star [8], or a subcubic [12,4] graph; (iii) each connected component of $G_\cap$ has a fixed embedding [13]; or (iv) G_1 and G_2 are biconnected and $G_\cap$ is connected [14].

For larger values of k, instead, both the SEFE and the PTBE-k problems are known to be $\mathcal{NP}$-complete. In fact, Gassner et al. [15] proved $\mathcal{NP}$-completeness of SEFE for $k \geq 3$, while Hoske [12] proved $\mathcal{NP}$-completeness of PTBE-k for k unbounded. On the other hand, if the embedding of the input graphs is fixed, SEFE becomes polynomial-time solvable for $k = 3$, but remains $\mathcal{NP}$-complete for $k \geq 14$ [16].

In Chapter 11 of the Handbook of Graph Drawing and Visualization [3], the SEFE problem with *sunflower intersection* (Sunflower SEFE) is reported as an open question (Open Problem 7). In this setting, the intersection graph $G_\cap$ is such that, if an edge belongs to $G_\cap$, then it belongs to all the input graphs. See Fig. 1(a) for an example. Note that every instance of SEFE with $k = 2$ obviously has sunflower intersection. We remark that the same technique used in [8] to prove that SEFE of two graphs with connected intersection is equivalent to PTBE-2 can be applied to prove that Sunflower SEFE of k graphs with connected intersection is equivalent to PTBE-k. Haeupler et al. [10] conjectured that Sunflower SEFE is polynomial-time solvable. However, Schaefer [4] recently proved that this problem is $\mathcal{NP}$-complete for $k \geq 3$ by providing a reduction from PTBE-k. Observe that, this reduction produces instances of Sunflower SEFE in which the intersection graph is a spanning forest composed of an unbounded number of star graphs [4].

1.1. Our results

In this paper, we prove that Sunflower SEFE is $\mathcal{NP}$-complete for $k \geq 3$ even if $G_\cap$ is a single spanning tree and all the input graphs are biconnected. We remark that having higher connectivity, both on the input graphs and on their intersection, is often a key factor to obtain polynomial-time solutions for this problem [8,10,11,14].

Given the equivalence between the connected version of Sunflower SEFE and PTBE-k [8], our result implies the $\mathcal{NP}$-completeness of PTBE-k for $k \geq 3$; however, the biconnectivity of the graphs in Sunflower SEFE is not maintained in the reduction to PTBE-k, that is, instances $\langle T, E_1, \dots, E_k \rangle$ produced by the reduction are such that graphs $G_i = T \cup E_i$ are possibly not biconnected. In this direction, we investigate the complexity of PTBE-k under stronger assumptions on the connectivity of the input graphs and show that it remains $\mathcal{NP}$-complete for $k \geq 3$ even if two of the input graphs G_i are biconnected. Further, we prove $\mathcal{NP}$-completeness for this problem when T is a star; this setting, in which the tree T basi-