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Algorithmic correspondence for intuitionistic modal mu-calculus ☆

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ABSTRACT

In the present paper, the algorithmic correspondence theory developed in Conradie and Palmigiano [9] is extended to mu-calculi with a non-classical base. We focus in particular on the language of bi-intuitionistic modal mu-calculus. We enhance the algorithm ALBA introduced in Conradie and Palmigiano [9] so as to guarantee its success on the class of recursive mu-inequalities, which we introduce in this paper. Key to the soundness of this enhancement are the order-theoretic properties of the algebraic interpretation of the fixed point operators. We show that, when restricted to the Boolean setting, the recursive mu-inequalities coincide with the "Sahlqvist mu-formulas" defined in van Benthem, Bezhanishvili and Hodkinson [22].

0. Introduction

Modal mu-calculus [17] is a logical framework combining simple modalities with fixed point operators, enriching the expressivity of modal logic so as to deal with infinite processes like recursion. It has a simple syntax, an easily given semantics, and is decidable. Modal mu-calculus has become a fundamental logical tool in theoretical computer science and has been extensively studied [3], and applied for instance in the context of temporal properties of systems, and of infinite properties of concurrent systems. Many expressive modal and temporal logics such as PDL, CTL, CTL* can be seen as fragments of the modal mu-calculus [3,15]. It provides a unifying framework connecting modal and temporal logics, automata theory and the theory of games, where fixed point constructions can be used to talk about the long term strategies of players, as discussed in [23].

Correspondence theory studies the relationships between classical first- and second-order logic, and modal logic, interpreted on Kripke frames. A modal and a first-order formula *correspond* if they define the same class of structures. Specifically, Sahlqvist theory is concerned with the identification of syntactically specified classes of modal formulas which correspond to first-order formulas. Sahlqvist-style frame-correspondence theory for modal mu-calculus has recently been developed in [22]. Such analysis strengthens the general mathematical theory of the mu-calculus, facilitates the transfer of results

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from first-order logic with fixed points, and aids in understanding the meaning of mu-formulas interpreted over frames, which is often difficult to grasp.

The correspondence results in [22] are developed purely model-theoretically. However, they can be naturally encompassed within the existing *algebraic* approach to correspondence theory [5,9,10], and generalized to mu-calculi on a weaker-than-classical (and, particularly, *intuitionistic*) base.

There are three types of reasons for studying (bi-)intuitionistic mu-calculi. Firstly, the correspondence results obtained in this setting project onto those obtainable in the classical setting of [22]. Conceptually, this means that the correspondence mechanisms for mu-calculi are intrinsically independent of their being set in classical logic, and hence the non-classical mu-calculi provide clearer insights into their nature, by abstracting from unneeded assumptions.[1] Secondly, these mu-calculi also bring practical advantages, since their greater generality means of course wider applicability. Finally, it can be argued that such a study is now timely, given that closely related areas of logic such as constructive modal logics and type theory are of increasing foundational and practical relevance in such fields as semantics of programming languages [12], and intuitionistic modal mu-calculi can be a valuable tool to these investigations.

Our contribution As motivated above, we work on a non-classical base, and our setting of choice is bi-intuitionistic modal logic [18,16]. Besides being interesting in its own right, the bi-intuitionistic modal logic also lends itself to certain types of more systematic analysis. For instance, it provides a methodologically useful setting for the order-theoretic analysis of correspondence theory. Indeed, its array of connectives is representative of the various order-theoretic behaviors one is likely to encounter, and hence provides a blueprint for transferring this analysis to other logics. Moreover, this analysis includes as a special case the correspondence theory for classical modal mu-calculus of [22]. Following the methodology developed in [10], the algebraic and order-theoretic principles underlying these results are isolated. This forms an intermediate level of analysis which is added to the model-theoretic analysis in [22].

In fact, this intermediate level of analysis makes it possible to recognize that even the lattice-distributivity plays no essential role for the crucial order-theoretic preservation properties of fixed points; accordingly, these properties are stated in the vastly more general setting of complete (not necessarily distributive) lattices, paving the way to the development of correspondence theory for substructural mu-calculi.

The fact that the intermediate level of analysis is conducted on algebras makes it possible to develop the crucial part of correspondence theory independently of the specific way the relational semantics is defined, given that different relational semantics can be associated with a given non-classical (fixed point) logic.

In the present paper, we extend the algorithm ALBA of [9] to the language of bi-intuitionistic modal mu-calculus. ALBA is an algorithm, based on a calculus of rewrite rules which, if successful, effectively calculates first-order correspondents for formulas of distributive, intuitionistic and classical modal logics. We define the class of recursive inequalities (see Definition 3.2) for the bi-intuitionistic modal mu-calculus which is the bi-intuitionistic counterpart of the Sahlqvist mu-formulas defined in [22]. We prove that the enhanced ALBA is successful on all recursive mu-inequalities, and hence that each of them has a frame correspondent in first-order logic with least fixed points (FO+LFP) [13].

It is worth stressing that all the results and in particular all the practical reductions developed for bi-intuitionistic modal mu-calculus are immediately applicable to the classical case.

Structure Within the preliminary section, Section 1.1 collects some details about the algebraic and relational semantics of bi-intuitionistic modal logic. In Section 1.2, algebraic-algorithmic correspondence theory is illustrated by means of an extensive example, and in Section 1.3 the calculus for correspondence for bi-intuitionistic modal logic is introduced and discussed. In Section 2, the stage is set for extending correspondence theory to mu-calculus: in Section 2.1, the relevant order-theoretic preservation properties of extremal fixed points are stated, in a general setting of complete (not necessarily distributive) lattices. In Section 2.2, the language and semantics of the bi-intuitionistic modal mu-calculus is introduced, together with the expanded language which facilitates the algebraic correspondence reductions. In Sections 2.3 and 2.4, the formal tools for the enhanced version of ALBA are introduced, in the form of so-called *approximation* and *adjunction rules* for fixed point binders, and the soundness of these rules is proven in terms of the order-theoretic properties of Section 2.1. In Section 2.5 the limitations of these rules are discussed, which motivates the developments of Sections 4 and 5. In Section 3, the recursive mu-inequalities are defined in the same uniform style discussed and advocated in [5], which pivots on the order-theoretic properties of the algebraic interpretation of the logical connectives. This class is compared with other Sahlqvist-type classes in the literature. In Section 4, we define certain syntactic shapes of formulas, the (*normal*) *inner formulas*, which guarantee the applicability of the approximation rules as stated in Section 2.3, and of a special rephrasing of the adjunction rules, given in Section 5. In Section 6, we show the execution of the algorithm on two examples. In Section 7, the relation between inner formulas and recursive inequalities is unfolded, which serves as a technical device in the proof that ALBA, augmented with the rules defined in the previous sections, is successful on all recursive inequalities.

[1] In particular, the bi-intuitionistic setting accounts for the projection over the classical setting more naturally than the intuitionistic one, for various technical reasons which will be expanded upon in Remark 3.6.