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A derivative for complex Lipschitz maps with generalised Cauchy–Riemann equations

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ABSTRACT

We introduce the Lipschitz derivative or the L-derivative of a locally Lipschitz complex map: it is a Scott continuous, compact and convex set-valued map that extends the classical derivative to the bigger class of locally Lipschitz maps and allows an extension of the fundamental theorem of calculus and a new generalisation of Cauchy–Riemann equations to these maps, which form a continuous Scott domain. We show that a complex Lipschitz map is analytic in an open set if and only if its L-derivative is a singleton at all points in the open set. The calculus of the L-derivative for sum, product and composition of maps is derived. The notion of contour integration is extended to Scott continuous, non-empty compact, convex valued functions on the complex plane, and by using the L-derivative, the fundamental theorem of contour integration is extended to these functions.

1. Introduction

Complex numbers and complex functions are used in a wide range of areas in engineering, physics as well as mathematics and their role in many fields such as quantum physics and quantum computation is fundamental and indispensable. Many programming languages have a pair of floating point numbers to represent complex numbers. In [31], Knuth proposed the Quarter-imaginary base $2i$, where $i = \sqrt{-1}$, as a system for computing basic arithmetic operations on complex numbers.

In more recent years, there has been a great deal of interesting work by the Blum–Shub–Smale (BSS) model community and also the type two theory (TTE) community in the computability of various subsets of the complex plane and complex mappings, in particular in connection with complex dynamical systems and fractal objects [3,29,30,8]. In [34,35], the computational complexity of Taylor series and the constructive aspects of analytic functions have been examined. Lately, effective analytical continuation and Riemann surfaces have been studied in [43] and a mixed BSS–TTE model has been used to study computability of analytic functions [25].

Our final goal is to develop a domain-theoretic data type for functions of a complex variable, including analytic maps, so that it can simultaneously represent the function and its differential properties. In this paper, we will tackle the task of formulating a derivative for an appropriate family of complex maps.

The class of complex Lipschitz maps, with their rich closure and convergence properties, provides us with a suitable class to develop a complex function data type. In fact, this class contains the fundamental class of piecewise linear maps, is closed under taking absolute value and the basic arithmetic operations on functions. Moreover, as well as being uniformly

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continuous, Lipschitz maps with uniformly bounded Lipschitz constants are also closed under convergence with respect to the sup norm. Vector real-valued Lipschitz maps are, by Rademacher's theorem, differentiable almost everywhere on finite dimensional Euclidean spaces [10, p. 148], and by Kirszbraun's theorem [24, p. 202] these maps can always be extended from any subset of the Euclidean space to the whole space. Furthermore, complex Lipschitz maps unify the two basic classes of analytic and anti-analytic maps.

The prominent properties of analytic maps are based on complex differentiation. The question therefore arises: how can we capture the differential properties of a complex Lipschitz map? In 1980's, Frank Clarke developed a set-valued derivative for real-valued Lipschitz maps on Euclidean spaces [9]. On finite dimensional Euclidean spaces, the Clarke gradient has non-empty compact and convex subsets of the Euclidean space as its values and it extends the classical derivative of C^1 maps to the bigger class of Lipschitz maps.

Motivated by structures in domain theory, the L-derivative of real Lipschitz maps and a domain for these maps were introduced for functions of a single real variable in [20] by generalising the notion of Lipschitz constant of a function in an open set to a non-empty compact real interval, which gives finitary information about the rate of growth of the function in the open set. The L-derivative of a map at a point is then obtained by collecting together all such finitary differential properties that the map satisfies in its neighbourhoods. This gives a shrinking sequence of non-empty compact sets, in the spirit of interval analysis [33], whose intersection is the L-derivative of the map at the given point. Subsequently, the L-derivative was extended to higher dimensional maps, first as a hyper-rectangular valued map [21], and later as the non-empty, convex and compact set-valued map, which was also defined on general Banach spaces. The L-derivative was then shown to be equivalent to the Clarke gradient for maps on finite dimensional Euclidean spaces [16]. Given that the L-derivative is built by collecting finitary differential properties of a map, it has been used in a number of areas in computer science. In particular, the domain of locally Lipschitz real-valued maps has been used to develop a PCF type programming language for computable and differentiable functions [11]. The question is whether we can do the same for complex Lipschitz maps.

In this paper, we define the Lipschitz derivative or L-derivative of a complex locally Lipschitz map, which is always defined as a non-empty, convex and compact set in the complex plane and is obtained from the set of finitary Lipschitz properties of the map. Compared to the case of real maps, the new complex derivative is formulated differently but is inspired from the real case in that we use a generalisation of the notion of Lipschitz constant this time to a non-empty compact convex subset of the complex plane. At any given point, the L-derivative of a complex Lipschitz map is obtained as the intersection of a shrinking sequence of non-empty convex compact sets, which are the generalised Lipschitz constants of the map in the neighbourhoods of the point and each of which gives a finitary description of the derivative. The L-derivative of a complex map is Scott continuous with respect to the Scott topology of the continuous Scott domain of the non-empty convex and compact subsets of the complex plane partially ordered by reverse inclusion. It extends the classical derivative of complex differentiable maps to complex Lipschitz maps such that for analytic maps the L-derivative at any point would be a singleton, namely the classical derivative at that point. Thus, the L-derivative of an analytic map will be a maximal element of the continuous Scott domain, which also maps every point to a maximal element. This therefore presents another example where a classical notion in mathematical analysis is captured as a subset of the maximal elements of a domain [39,14,13,15,32,18,19].

The L-derivative provides us with a fundamental theorem of calculus for complex Lipschitz maps, a duality between Scott continuous compact, convex valued maps and their Lipschitz primitives. As analytic functions and anti-analytic functions (such as conjugation) are both Lipschitz and anti-analytic maps are non-differentiable, the L-derivative also gives rise to a unifying derivative for both of these fundamental classes.

The calculus of the L-derivative is derived for sum and product; the chain rule for the L-derivative of composition of functions is also developed. We provide some simple examples of the L-derivative of basic maps like conjugation and absolute value that are not differentiable.

We then use the directional derivative of a complex map [27] to define a differential transformation which converts the differential properties of a planar vector valued Lipschitz map into the differential properties of the induced complex map, whose real and imaginary parts are given by the vector valued map. This transformation gives what we call the C-derivative of the complex function by mapping the Clarke gradient of the vector function obtained using the real and imaginary parts of the complex map from a non-empty compact convex subset of the vector space of 2×2 real matrices into the complex plane. We then show that the L-derivative and the C-derivative of a complex Lipschitz map are equal as non-empty compact convex subsets, which gives a set-valued generalisation of the Cauchy–Riemann equations to complex Lipschitz maps. It shows that the L-derivative is the convex hull of a union of disks in the complex plane. For analytic functions, this union is reduced to a single point and we obtain the classical Cauchy–Riemann equations. The extension of Cauchy–Riemann equations to Lipschitz maps provides a common framework to study the differential properties of analytic and anti-analytic maps. In particular we show that the L-derivative of the conjugation map is the unit disk centred at the origin. All these results are extended to functions of several complex variables and thus present a new view on complex differentiation.

Since they were used in the theory of functions by Cauchy and Riemann in the 19th century [40], Cauchy–Riemann equations have been at the basis of Harmonic analysis and Laplace equations with applications in partial differential equations. In the context of the current generalisation of these equations, we are in particular interested in the approximation of analytic maps by piecewise linear functions and construction of a domain and a data type for analytic functions.