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Random product of substitutions with the same incidence matrix

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ABSTRACT

Any infinite sequence of substitutions with the same matrix of the Pisot type defines a symbolic dynamical system which is minimal. We prove that, to any such sequence, we can associate a compact set (Rauzy fractal) by projection of the stepped line associated with an element of the symbolic system on the contracting space of the matrix. We show that this Rauzy fractal depends continuously on the sequence of substitutions, and investigate some of its properties; in some cases, this construction gives a geometric model for the symbolic dynamical system.

1. Introduction

It is well known that any unimodular irreducible Pisot substitution defines a dynamical system that is a finite extension of a toral translation, and it is conjectured that such a dynamical system is in fact measurably equivalent to a toral translation, see [8,10]. A geometric model of the symbolic system can be obtained by projecting the discrete line associated to a fixed point of the substitution along its asymptotic direction, as we explain more precisely in the next section.

In some cases, this can be extended to systems generated by an infinite sequence of substitutions belonging to a finite set $\mathcal{S}$ (so-called $\mathcal{S}$-adic systems), the best example being that of the Sturmian sequences, which are almost 1–1 extensions of irrational circle rotations, the adic expansion being given by the additive continuous fraction expansion of the angle.

It would be very interesting to be able to generalize this property; however, in the general case, we cannot expect it to hold for all sequences: already, one can check that the closure of the set of all Sturmian sequences contains periodic sequences, with bounded complexity and finite $\mathcal{S}$-adic expansion; it is a degenerate case, where the symbolic model is finite. The existence of non-balanced episturmian sequences (see [6]) is another obstruction; in that case, the classical construction of the Rauzy fractal by projection cannot work, since the projected set is well defined, but not bounded. The existence of minimal, but not uniquely ergodic $\mathcal{S}$-adic systems associated with interval exchange maps is a third one: for the corresponding symbolic system, the frequency is not defined, hence the discrete line (see below) associated with a symbolic sequence has no asymptotic direction, so the projection is not even defined.

In this paper, we solve the problem in a restricted case: we consider a matrix A with positive integer coefficients of the Pisot type (that is, all eigenvalues are nonzero, and all eigenvalues except one are strictly smaller than 1 in modulus),

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and a finite set $\mathcal{S} = \{\sigma_1, \dots, \sigma_k\}$ of substitutions with the same matrix A. We first prove that any infinite sequence of elements of $\mathcal{S}$ defines a minimal symbolic system, by defining a generalized fixed point (the *limit point*). We then prove, using a generalized prefix–suffix expansion, that this limit point stays within bounded distance of the expanding line of the matrix, and that we can associate a compact set by projection of this limit point on the contracting space. We also prove, by using a generalization of the classical IFS theorem of Hutchinson [9], that this set depends continuously on the sequence of substitutions, and investigate some of its properties.

In Section 2, we fix the notations, and show that a primitive sequence of substitutions on an alphabet $\mathcal{A}$ defines a minimal dynamical system in $\mathcal{A}^{\mathbb{N}}$. In Section 3, we prove that if all the substitutions have the same incidence matrix of the Pisot type, we can generalize to that case the projection construction of the Rauzy fractal. In Section 4, we generalize the classical iterated function system to the case of an infinite sequence of contractions, and apply it to our case to prove that the generalized Rauzy fractal depends continuously on the sequence of substitutions. In Section 5, we give a few examples, and make some remarks.

2. Substitutions and adic systems

2.1. General setting

Let $\mathcal{A} := \{1, \dots, d\}$ be a finite set of cardinality $d > 1$, called the alphabet. We denote by $\mathcal{A}^*$ the free monoid on $\mathcal{A}$ (set of finite words on the alphabet $\mathcal{A}$, with empty word denoted by ε, endowed with the concatenation map). We denote by $\mathcal{A}^{\mathbb{N}}$ the set of infinite sequences on $\mathcal{A}$, with the natural product topology.

The length $|w|$ of a word $w \in \mathcal{A}^n$ with $n \in \mathbb{N}$ is defined as $|w| = n$. For any letter $a \in \mathcal{A}$, we denote the number of occurrences of a in $w = w_1 w_2 \dots w_{n-1} w_n$ by $|w|_a = \sharp\{i \mid w_i = a\}$. We will denote by $l : \mathcal{A}^* \mapsto \mathbb{N}^d : w \mapsto (|w|_a)_{a\in\mathcal{A}} \in \mathbb{N}^d$ the natural homomorphism (abelianization map) obtained by abelianization of the free monoid.

A substitution over the alphabet $\mathcal{A}$ is a nonerasing endomorphism of the free monoid $\mathcal{A}^*$. To any substitution σ, one can associate its incidence matrix M, which is the $d \times d$ matrix obtained by abelianization, i.e. $M_{i,j} = |\sigma(j)|_i$. By construction, one has $l(\sigma(w)) = Ml(w)$ for any word $w \in \mathcal{A}^*$.

Definition 2.1. A substitution σ is *primitive* if there exists an integer k such that, for each pair $(a, b) \in \mathcal{A}^2$, $|\sigma^k(a)|_b > 0$.

It is equivalent to suppose that the incidence matrix is primitive, that is, this matrix has a strictly positive power. We will always suppose that the substitution is primitive, this implies that for all letter $j \in \mathcal{A}$ the length of the successive iterations $\sigma^k(j)$ tends to infinity.

Remark 1. Since the incidence matrix of a primitive substitution is a primitive matrix, by the Perron–Frobenius theorem, it has a simple real positive dominant eigenvalue β.

2.2. Dynamical system defined by a primitive substitution

A substitution σ naturally extends to the set $\mathcal{A}^{\mathbb{N}}$ of infinite sequences with value in $\mathcal{A}$, by defining $\sigma(u_1 u_2 \dots) = \sigma(u_1)\sigma(u_2)\dots$. We say that a sequence $u \in \mathcal{A}^{\mathbb{N}}$ is a periodic point of σ if there exists some integer k such that $\sigma^k(u) = u$. One easily proves that any primitive substitution has periodic points, and that two periodic points with the same initial letters are equal, hence there are at most d periodic points.

We denote by S the shift on $\mathcal{A}^{\mathbb{N}}$, defined by $S(u) = v$, where v is the sequence such that, for all $i \in \mathbb{N}$, $v_i = u_{i+1}$.

Definition 2.2. Let σ be a primitive substitution, and let u be a periodic point of σ. Let $\Omega_\sigma = \overline{\{S^n(u) \mid n \in \mathbb{N}\}}$ be the closure of the orbit of u by the shift.

The dynamical system defined by the substitution σ is (Ω_σ, S).

Remark 2. The primitivity condition implies that any word occurring in a periodic word of σ also appears in any other periodic word of σ, hence all these periodic words have same closure for the shift, and the set Ω_σ does not depend on the particular periodic word we have chosen, only on the substitution σ.

2.3. Pisot substitutions

We recall a classical definition of algebraic number theory:

Definition 2.3. A Pisot number is an algebraic integer $\beta > 1$ such that each Galois conjugate $\beta^{(i)}$ of β satisfies $|\beta^{(i)}| < 1$.

By analogy, we define substitutions of Pisot type and irreducible substitutions of Pisot type: