Theoretical Computer Science 524 (2014) 27–33

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Complexity results for rainbow matchings

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ARTICLE INFO

Article history:
Received 13 December 2012
Accepted 16 December 2013
Communicated by T. Erlebach

Keywords:
Rainbow matching
Computational complexity
NP-completeness
APX-completeness
Parameterized complexity

ABSTRACT

A rainbow matching in an edge-colored graph is a matching whose edges have distinct colors. We address the complexity issue of the following problem, MAX RAINBOW MATCHING: Given an edge-colored graph G, how large is the largest rainbow matching in G? We present several sharp contrasts in the complexity of this problem.
We show, among others, that

- MAX RAINBOW MATCHING can be approximated by a polynomial algorithm with approximation ratio $2/3 - \varepsilon$.
- MAX RAINBOW MATCHING is APX-complete, even when restricted to properly edge-colored linear forests without a 5-vertex path, and is solvable in polynomial time for edge-colored forests without a 4-vertex path.
- MAX RAINBOW MATCHING is APX-complete, even when restricted to properly edge-colored trees without an 8-vertex path, and is solvable in polynomial time for edge-colored trees without a 7-vertex path.
- MAX RAINBOW MATCHING is APX-complete, even when restricted to properly edge-colored paths.

These results provide a dichotomy theorem for the complexity of the problem on forests and trees in terms of forbidding paths. The latter is somewhat surprising, since, to the best of our knowledge, no (unweighted) graph problem prior to our result is known to be NP-hard for simple paths.
We also address the parameterized complexity of the problem.

1. Introduction and results

Given a graph $G = (E(G), V(G))$, an edge coloring is a function $\phi : E(G) \to \mathcal{C}$ mapping each edge $e \in E(G)$ to a color $\phi(e) \in \mathcal{C}$; ϕ is a *proper* edge-coloring if, for all distinct edges e and e', $\phi(e) \neq \phi(e')$ whenever e and e' have an endvertex in common. A (properly) edge-colored graph (G, ϕ) is a pair of a graph together with a (proper) edge coloring. A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. Rainbow subgraphs appear frequently in the literature, for a recent survey we point to [13].

In this paper we are concerned with *rainbow matchings*, i.e., matchings whose edges have distinct colors. One motivation to look at rainbow matchings is Ryser's famous conjecture from [20], which states that every Latin square of odd order contains a Latin transversal. Equivalently, the conjecture says that every proper edge coloring of the complete bipartite graph $K_{2n+1,2n+1}$ with $2n+1$ colors contains a rainbow matching with $2n+1$ edges.

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0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.12.013

One often asks for the size of the largest rainbow matching in an edge-colored graph with certain restrictions (see, e.g., [4,14,15,22]). In this paper, we consider the complexity of this problem. In particular, we consider the complexity of the following two problems, and we will restrict them to certain graph classes and edge colorings.

RAINBOW MATCHING

Instance: Graph G with an edge-coloring and an integer k.
Question: Does G have a rainbow matching with at least k edges?

RAINBOW MATCHING is also called MULTIPLE CHOICE MATCHING in [8, Problem GT55]. The optimization version of the decision problem RAINBOW MATCHING is:

MAX RAINBOW MATCHING

Instance: Graph G with an edge-coloring.
Output: A largest rainbow matching in G.

Only the following complexity result for RAINBOW MATCHING is known. Note that the considered graphs in the proof are not properly edge colored. When restricted to properly edge-colored graphs, no complexity result is known prior to this work.

Theorem 1. *(See [12].)* RAINBOW MATCHING *is NP-complete, even when restricted to edge-colored bipartite graphs.*

In this paper, we analyze classes of graphs for which MAX RAINBOW MATCHING can be solved and thus RAINBOW MATCHING can be decided in polynomial time, or is NP-hard. Our results are:

- There is a polynomial-time $(2/3-\varepsilon)$-approximation algorithm for MAX RAINBOW MATCHING for every $\varepsilon > 0$.
- MAX RAINBOW MATCHING is APX-complete, and thus RAINBOW MATCHING is NP-complete, even for very restricted graphs classes such as
  – edge-colored complete graphs,
  – properly edge-colored paths,
  – properly edge-colored P_8-free trees in which every color is used at most twice,
  – properly edge-colored P_5-free linear forests in which every color is used at most twice,
  – properly edge-colored P_4-free bipartite graphs in which every color is used at most twice.

  These results significantly improve Theorem 1. We also provide an inapproximability bound for each of the listed graph classes.
- MAX RAINBOW MATCHING is solvable in time $O(m^{3/2})$ for m-edge graphs without P_4 (induced or not); in particular for P_4-free forests.
- MAX RAINBOW MATCHING is polynomially solvable for P_7-free forests with bounded number of components; in particular for P_7-free trees.
- MAX RAINBOW MATCHING is fixed parameter tractable for P_5-free forests, when parameterized by the number of the components.

The next section contains some relevant notation and definitions. Section 3 deals with approximability and inapproximability results, Section 4 discusses some polynomially solvable cases, and Section 5 addresses the parameterized complexity. We conclude the paper in Section 6 with some open problems.

2. Definitions and preliminaries

We consider only finite, simple, and undirected graphs. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. An edge xy of a graph G is a *bridge* if $G - xy$ has more components than G. If G does not contain an induced subgraph isomorphic to another graph F, then G is *F-free*.

For $\ell \geqslant 1$, let P_ℓ denote a chordless path with ℓ vertices and $\ell - 1$ edges, and for $\ell \geqslant 3$, let C_ℓ denote a chordless cycle with ℓ vertices and ℓ edges. A *triangle* is a C_3. For $p, q \geqslant 1$, $K_{p,q}$ denotes the complete bipartite graph with p vertices of one color class and q vertices of the second color class; a *star* is a $K_{1,q}$. A complete graph with p vertices is denoted by K_p; $K_p - e$ is the graph obtained from K_p by deleting one edge. An *r-regular graph* is one in which each vertex has degree exactly r. A forest in which each component is a path is a *linear forest*.

The *line graph* $L(G)$ of a graph G has vertex set $E(G)$, and two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident. By definition, every matching in G corresponds to an independent set in $L(G)$ of the same size, and vice versa. One of the main tools we use in discussing rainbow matchings is the following concept that generalizes line graphs naturally:

Definition 1. The *color-line graph* $CL(G)$ of an edge-colored graph G has vertex set $E(G)$, and two vertices in $CL(G)$ are adjacent if the corresponding edges in G are incident or have the same color.