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Blockers and antiblockers of stable matchings

Pavlos Eirinakis [a,*], Dimitrios Magos [b], Ioannis Mourtos [a]

[a] Department of Management Science and Technology, Athens University of Economics and Business, 76 Patission Ave., 104 34 Athens, Greece
[b] Department of Informatics, Technological Educational Institute of Athens, Ag. Spyridonos Str., 12210 Egaleo, Greece

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ABSTRACT

An implicit linear description of the stable matching polytope is provided in terms of the blocker and antiblocker sets of constraints of the matroid-kernel polytope. The explicit identification of both these sets is based on a partition of the stable pairs in which each agent participates. Here, we expose the relation of such a partition to rotations. We provide a time-optimal algorithm for obtaining such a partition and establish some new related results; most importantly, that this partition is unique.

1. Introduction

The *Stable Matching* problem was introduced by Gale and Shapley in [20] to model the process of matching the agents of two distinct sets in a stable fashion; the resulting matching is *stable* in the sense that, with respect to the agents' preferences, there exists no (acceptable) pair such that both of its members would prefer to be matched with each other than with (one of) their current pairs in that matching.

The stable matching framework has been utilized to model a plethora of real-life problems, such as assigning applicants to colleges [27,30,8], students to schools [8,1–3], residents to hospitals [31,32,36,33,35,34] students to sororities [28], and cadets to military branches [40]. The common ground of all these problems is that there exists a centralized stable matching scheme, which assigns applicants to posts; each applicant is assigned to at most one post, while each post can admit a predetermined number of applicants (also known as its *quota*). This 'many-to-one' version is called *Stable Admissions* (*SA*). Despite its wide applicability, several cases cannot be handled with this framework; in fact, the existence of even one applicant who can hold more than one posts may change the matching outcome substantially (see Example 2.2 in [11]). Hence, any matching market in which the agents of both sets can be matched with more than one of the agents of the opposite set should be handled in terms of the *many-to-many Stable Matching* (*MM*) problem; a well-known example is the medical interns' market in the UK [36].

The stable matching problem that is mostly explored from an algorithmic perspective [20,25,21,26,22,36] is *Stable Marriage* (*SM*), i.e., the simplified version where the two sets of agents correspond to (monogamous) men and women. With respect to the SA problem, a deferred-acceptance algorithm that always finds a solution is provided in [20]. An extended version of that algorithm (known as the Extended Gale–Shapley (EGS) algorithm) is presented in [22]. The EGS algorithm also identifies some (but not all) non-stable pairs, i.e., pairs of agents that do not participate in any solution. An algorithm identifying all non-stable pairs was given in [12]. In the MM context, the EGS algorithm is generalized in [6]. The notion

* Corresponding author. Tel.: +30 210 8203731.
E-mail addresses: peir@aueb.gr (P. Eirinakis), dmagos@teiath.gr (D. Magos), mourtos@aueb.gr (I. Mourtos).

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of *rotations* (i.e., a 'mechanism' that allows moving from one solution to another) and their partial order (known as the *rotation-poset*) is introduced in [9]; there, rotations are utilized to provide algorithms for identifying the egalitarian and the 'optimal' solution. An algorithm finding all stable pairs (i.e., pairs that participate in at least one solution) and all stable assignments (i.e., maximal groups of firms or workers simultaneously assigned to a single agent in a solution) is presented in [13,14]. [14] also revisits the definition of the rotation-poset, and provides algorithms for finding the minimum-weight solution and enumerating all solutions of a given instance. An algorithm for finding the minimum-regret solution is presented in [15]. Rotations have also been utilized in the more generic *Stable Allocation* (SL) problem, i.e., a many-to-many framework in which quotas need not be integer, while the value allocated from an agent of one set to an agent of the other set can be fractional and greater than 1. Hence, several interesting structural results on rotations obtained in the SL setting can be readily applied to the MM case [10].

From a polyhedral perspective, work on the SA and the MM problem remains limited, especially when compared to that of the SM [42,38,37,4,5,29,41,18,16]. A linear description of the SA polytope is given by Baïou and Balinski in [7] and is used in establishing the geometry of fractional SA solutions in [39]. Moreover, it is the basis of the linear description of the MM problem presented in [24] as a special case of the *laminar classified stable matching* polytope (through cloning of the agents of one side). The Baïou–Balinski approach includes an exponential number of constraints (and thus is accompanied by a separation algorithm). On the other hand, Fleiner [19] provides a polynomially-sized linear description of the MM (and hence of the SA) polytope; that description utilizes his characterization of the *matroid-kernel* polytope, which is based on the theory of blocking polyhedra and lattice polyhedra [17,18].

More specifically, Fleiner shows that *ordered matroids* [17,18] generalize the stable matching problem. In that regard, he provides an implicit linear description of the stable matching polytope in terms of the blocker and the antiblocker sets of constraints of the corresponding matroid-kernel polytope [17]. Note that this characterization, although implicit, is generic; it is a unified description of the SM, SA, and MM polytopes. An explicit description of these polytopes can be derived by specifying the blocker and the antiblocker sets of constraints. In the SM case, this linear description coincides with the one provided in [38]. In the MM (and hence the SA) case, the construction of both such sets is based on a partition of the pairs $E(z)$, in which each agent z participates, into q_z components $E_i(z)$, $i = 1, \dots, q_z$ (where q_z is the quota of z), such that, for any i, at most one element of $E_i(z)$ participates in any MM solution [19]. The same partitioning notion (discussed there in terms of "bin-packing") is the key concept of the analysis presented in [39] in the SA context. Let us use the term *Stable Pairs Partition* (*SPP*) to refer to such a partition, since it can be restricted only to stable pairs; non-stable pairs do not participate in any MM solution by definition. Note that an SPP always exists; an iterative algorithm for finding one is provided in [19].

In this paper, we expose the relation of SPP to rotations. Hence, we establish that rotations can accommodate the polyhedral description of the MM polytope. Moreover, we provide a time-optimal algorithm for obtaining an SPP; since this procedure is a necessary pre-processing step for applying the linear description of Fleiner, our proposed algorithm improves the efficiency of any process solving the SA or the MM problem that utilizes this linear description. Furthermore, we generalize a known theorem in the MM context and provide some interesting new results related to the SPP; most importantly, that it is unique.

The rest of the paper is organized as follows. Section 2 details basic notions on the Stable Matching problem. Section 3 presents the algorithm partitioning the set of stable pairs as well as structural results on the SPP. Note that we will use MM as our reference problem, since it generalizes both SA and SM, and hence any obtained results can be straightforwardly applied to them.

2. The MM problem

The MM problem is naturally defined in the context of a job market containing a set of workers and a set of firms in which each firm can hire a group of workers and each worker can be employed by multiple firms, so as to achieve stability under given preferences [36]. More specifically, let W denote the set of workers, F denote the set of firms, and $n = \max\{|W|, |F|\}$. Each agent (worker or firm) has preferences over the agents of the other set. These preferences are strict and transitive, hence they can be represented by ordered *preference lists*. Let $P(w)$ ($P(f)$) denote the preference list of worker w (firm f). Moreover, let '$>_w$' ('$>_f$') represent the preference operator of worker w (firm f). That is, $f_i >_w f_j$ denotes that worker w prefers working for f_i to f_j. Similarly, $w_i >_f w_j$ denotes that firm f prefers hiring w_i to w_j. The maximum number of workers a firm $f \in F$ can employ is given by an integer-valued function, i.e., its quota q_f. Similarly, q_w represents the maximum number of firms a worker $w \in W$ can work for. Preference lists may be incomplete. The absence of a firm f from worker w's preference list represents that w would prefer to leave a workspace empty than to work for f (and f is said to be unacceptable by w). The situation for firms is analogous. On the other hand, (w, f) is *acceptable* if f is included in $P(w)$ and w is included in $P(f)$. Under this setting, a solution to the MM problem, i.e., a stable matching of firms to workers, is defined as follows [6].

Definition 1 *(Stable matching).* A (maximal) set $M \subseteq W \times F$ is called a stable matching if (i) the quota of each agent is not exceeded, (ii) all agents are individually rational (i.e., not matched with an unacceptable partner), and (iii) there is no blocking pair, i.e., no pair $(w, f) \notin M$ such that w does not fulfill her quota in M or $f >_w f'$ with $(w, f') \in M$, and f does not fulfill its quota in M or $w >_f w'$ with $(w', f) \in M$.