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The cover times of random walks on random uniform hypergraphs

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ABSTRACT

Random walks in graphs have been applied to various network exploration and network maintenance problems. In some applications, however, it may be more natural, and more accurate, to model the underlying network not as a graph but as a hypergraph, and solutions based on random walks require a notion of random walks in hypergraphs. At each step, a random walk on a hypergraph moves from its current position v to a random vertex in a randomly selected hyperedge containing v. We consider two definitions of cover time of a hypergraph H. If the walk sees only the vertices it moves between, then the usual definition of cover time, $C(H)$, applies. If the walk sees the complete edge during the transition, then an alternative definition of cover time, the inform time $I(H)$ is used. The notion of inform time models passive listening which fits the following types of situations. The particle is a rumour passing between friends, which is overheard by other friends present in the group at the same time. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, but all receivers within the transmission range can hear.

In this paper we give an expression for $C(H)$ which is tractable for many classes of hypergraphs, and calculate $C(H)$ and $I(H)$ exactly for random r-regular, s-uniform hypergraphs. We find that for such hypergraphs, **whp**, $C(H)/I(H) \sim s(r-1)/r$, if $rs = O((\log \log n)^{1-\epsilon})$. For random r-regular, s-uniform multi-hypergraphs, constant $r \geq 2$, and $3 \leq s \leq O(n^{\epsilon})$, we also prove that, **whp**, $I(H) = O((n/s) \log n)$, i.e. the inform time decreases directly with the edge size s.

1. Introduction

The idea of a random walk on a hypergraph is a natural one. The particle making the walk picks a random edge incident with the current vertex. The particle enters the edge, and exits via a random endpoint, other than the vertex of entry. Two alternative definitions of cover time are possible for this walk. Either the particle sees only the vertices it visits, or it inspects all vertices of the hyperedge during the transition across the edge.

A random walk on a hypergraph models the following process. The vertices of a network are associated into groups, and these groups define the edges of the network. In the simplest case, the network is a graph so the groups are exactly the edges of the graph. In general, the groups may be larger, and represent friends, a family, a local computer network, or all receivers within transmission range of a directed transmission in an ad-hoc network. In this case the network is modelled as a hypergraph, the hyperedges being the group relationships. An individual vertex can be in many groups, and two vertices

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are neighbours if they share a common hyperedge. Within the network a particle (message, rumour, infection, etc.) is moving randomly from vertex to neighbouring vertex. When this transition occurs all vertices in a given group are somehow affected (infected, informed) by the passage of the particle *within the group*. Examples of this type of process include the following. The particle is an infection passed from person to person and other family members also become infected with some probability. The particle is a virus travelling on a network connection in an intra-net. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, and all receivers within the transmission range can hear. The particle is a rumour passing between friends, which may be overheard by other friends present in the group at the same time.

Let $H = (V(H), E(H))$ be a hypergraph. For $v \in V = V(H)$ let $d(v)$ be the degree of v, i.e. the number of edges $e \in E = E(H)$ incident with v, and let $d(H) = \sum_{v \in V} d(v)$ be the total degree of H. For $e \in E$, let $|e|$ be the size of hyperedge e, i.e. the number of vertices $v \in e$, respecting multiplicity. Let $N(v)$ be the neighbour set of v, $N(v) = \{w \in V : \exists e \in E, e \supseteq \{v, w\}\}$. We regard $N(v)$ as a multi-set in which each $w \in N(v)$ has a multiplicity equal to the number of edges e containing both v and w. A hypergraph is r regular if each vertex is in r edges, and is s-uniform if every edge is of size s. A hypergraph is simple if no edge contains a repeated vertex, and no two edges are identical. We assume a particle or message originated at some vertex u and, at step t, is moving randomly from a vertex v to a vertex w in $N(v)$. We model the problem conceptually as a random walk $\mathcal{W}_u = (W_u(0), W_u(1), \ldots, W_u(t), \ldots)$ on the vertex set of hypergraph H, where $W_u(0) = u$, $W_u(t) = v$ and $W_u(t+1) = w \in N(v)$.

Several models arise for reversible random walks on hypergraphs. Assume that the walk $\mathcal{W}$ is at vertex v, and consider the transition from that vertex. In the first model (Model 1), an edge e incident with v is chosen proportional to $|e| - 1$. The walk then moves to a random endpoint of that edge, other than v. This is equivalent to v choosing a neighbour w u.a.r. (uniformly at random) from $N(v)$, where vertex w is chosen according to its multiplicity in $N(v)$. The stationary distribution of v in Model 1 is given by

$$\pi_v = \frac{\sum_{e:v\in e}(|e|-1)}{\sum_{e\in E(H)}|e|(|e|-1)}.$$

In the case of graphs this reduces to $\pi_v = d(v)/2m$, where m is the number of edges in the graph. Alternatively (Model 2) when $\mathcal{W}$ is at v, edge e is chosen u.a.r. from the hyperedges incident with v, and *then* w is chosen u.a.r. from the vertices $w \in e$, $w \neq v$. The stationary distribution of v in Model 2 is given by

$$\pi_v = \frac{d(v)}{\sum_{u\in V(H)} d(u)},$$

which corresponds to the familiar formula for graphs. If the hypergraph is uniform (all edges have the same size) then the models are equivalent.

Random walks on graphs are a well studied topic, for an overview see e.g. [1,11]. Random walks on hypergraphs were used in [5] to cluster together electronic components which are near in graph distance for physical layout in circuit design. For that application, edges were chosen inversely proportional to their size, and then a random vertex within the edge was selected. A random walk model is also used for generalised clustering in [13]. As before, the aim is to partition the vertex set, and this is done via the Laplacian of the transition matrix. This technique has applications in data mining (see [10]) and clustering images from the www (see [15] and references therein). The paper [3] directly considers notions of cover time for random walks on hypergraphs, using Model 2. A further discussion of [3] is given below.

For a hypergraph H, we define the (vertex) cover time $C(H)$, the edge cover time $C_E(H)$, and the inform time $I(H)$. The (vertex) cover time $C(H) = \max_u C_u(H)$, where $C_u(H)$ is the expected time for the walk $\mathcal{W}_u$ to visit all vertices of H. Similarly, the edge cover time $C_E(H) = \max_u C_{u,E}(H)$, where $C_{u,E}(H)$ is the expected time to visit all hyperedges starting at vertex u.

Suppose that the walk $\mathcal{W}_u$ is at vertex v. Using e.g. Model 2, the walk first selects an edge e incident with v and then makes a transition to $w \in e$. The vertices of e are said to be *informed* by this move. The *inform time* $I(H)$, introduced in [3] as the *radio cover time*, is the maximum over start vertices u, of the expected time at which all vertices of the graph are informed. More formally, let $\mathcal{W}_u(t) = (W_u(0), W_u(1), \ldots, W_u(t))$ be the trajectory of the walk. Let $e(j)$ be the edge used for the transition from $W(j)$ to $W(j+1)$ at step j. Let $\mathcal{S}_u(t) = \cup_{j=0}^{t-1} e(j)$ be the set of vertices spanned by the edges of $\mathcal{W}_u(t)$. Let $\boldsymbol{I}_u$ be the step t at which $\mathcal{S}_u(t) = V$ for the first time, and let $I(H) = \max_u \mathbf{E}(\boldsymbol{I}_u)$. We use the name "inform time" rather than "radio cover time" in [3] to indicate the relevance of this term beyond the radio networks.

Several upper bounds on the cover time $C(H)$ are readily obtainable, for example an analogue of the $O(nm)$ bound for graphs [2] based on a twice round the spanning tree argument. For Model 1, replace each edge e by a clique of size $\binom{|e|}{2}$ to obtain an upper bound of $O(nm\overline{s^2})$ for connected hypergraphs. Here $\overline{s^2}$ is the expected squared edge size $(\sum_{e\in E(H)} |e|^2)/m$. Thus $C(H) = O(n^3 m)$. A better bound of $O(nm\bar{s}) = O(n^2 m)$ was shown in [3] for Model 2.

Similarly, a Matthews type bound of $O(\log n \cdot \max_{u,v} \mathbf{E}(\boldsymbol{H}_{u,v}))$ on the cover time exists, where $\mathbf{E}(\boldsymbol{H}_{u,v})$ is the expected hitting time of v starting from u. We contribute a bound on the cover time of a hypergraph given in Theorem 1, which allows