Theoretical Computer Science 491 (2013) 71–77

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Strong matching preclusion for augmented cubes

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ARTICLE INFO

Article history:
Received 7 November 2012
Received in revised form 15 March 2013
Accepted 2 May 2013
Communicated by S.-y. Hsieh

Keywords:
Interconnection networks
Perfect matching
Augmented cubes

ABSTRACT

The strong matching preclusion number of a graph is the minimum number of vertices and edges whose deletion results in a graph that has neither perfect matchings nor almost perfect matchings. The concept was introduced by Park and Son. In this paper, we study the strong matching preclusion problem for the augmented cube graphs. As a result, we find $\mathrm{smp}(AQ_n)$ and classify all optimal solutions.

1. Introduction

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An *almost perfect matching* in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. If a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Any such optimal set is called an *optimal matching preclusion set*. If G has neither a perfect matching nor an almost perfect matching, then $\mathrm{mp}(G) = 0$. This concept of matching preclusion was introduced by [1] and further studied by [4–10,23,24,26]. They introduced this concept as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [1] for details and additional references. In [25], the concept of strong matching preclusion was introduced. The *strong matching preclusion number* of a graph G, denoted by $\mathrm{smp}(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Any such optimal set is called an *optimal strong matching preclusion set*.

Useful distributed processor architectures offer the advantages of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. Such system topology forms the interconnection network. We refer the readers to [16] for recent progress in this area and the references in its extensive bibliography. In certain applications, every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [1]. Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets and optimal strong matching preclusion sets are those whose deletion gives an isolated vertex in

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0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.05.002

the resulting graph. Since interconnection networks are usually even, we only consider even graphs in this paper, that is, graphs with even numbers of vertices.

Proposition 1.1. *Let G be a graph with an even number of vertices. Then $smp(G) \leq mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proof. Since G is even, $\mathrm{mp}(G)$ is the minimum number of edges whose deletion leaves a graph with no perfect matchings. Since deleting all edges incident to a single vertex will give a graph with no perfect matchings, $\mathrm{mp}(G) \leq \delta(G)$. The claim $\mathrm{smp}(G) \leq \mathrm{mp}(G)$ is obviously true as every matching preclusion set is a strong matching preclusion set. □

An optimal solution of the form given in the proof of Proposition 1.1 is a *trivial (optimal) matching preclusion set*. Let F be an optimal strong matching preclusion set of a graph $G = (V, E)$. Suppose $F = F_V \cup F_E$ where F_V consists of vertices and F_E consists of edges. We may assume that no element in F_E is incident to an element in F_V since F is optimal. (If $e \in F_E$ is incident to an element of F_V, then $G - F = G - (F - \{e\})$.) We call F a *basic (optimal) strong matching preclusion set* if F is an optimal strong matching preclusion set of G and $G - F$ has an isolated vertex, that is, there exists a vertex v such that every vertex in F_V is a neighbor of v and every edge in F_E is incident to v. This includes the following scenario: F is a basic optimal matching preclusion set and $G - F$ is odd without almost perfect matchings. We can further restrict this class as follows. If $G - F$ is even and there is a vertex v such that every vertex in F_V is a neighbor of v and every edge in F_E is incident to v, then F is a *trivial (optimal) strong matching preclusion set*. For r-regular even graphs we have the following relationship between these classes of preclusion sets.

Proposition 1.2 (*[2]*). *Let $r \geq 2$. Let G be an r-regular even graph. Suppose that $smp(G) = r$. Then every basic strong matching preclusion set is trivial.*

Hypercubes are the most basic class of interconnection networks. However, they have shortcomings and a number of their variants were introduced to address some of the issues. One such popular variant is the class of augmented cubes introduced in [11]. As an improvement upon the hypercubes, the augmented cube graphs are designed to be superior in many aspects. Not only do they retain some of the favorable properties of the hypercubes but also possess some embedding properties that the hypercubes do not have. For instance, a hypercube of the nth dimension contains cycles of all lengths from 3 to 2^n whereas the hypercube contains only even cycles. As shown in [25], bipartite graphs are poor interconnection networks with respect to the strong matching preclusion property. However, augmented cubes are not bipartite and we will show in this paper that they have good strong matching preclusion properties.

We now define the n-dimensional augmented cube AQ_n as follows. Let $n \geq 1$, the graph AQ_n has 2^n vertices, each labeled by an n-bit binary string $u_1u_2 \cdots u_n$ such that $u_i \in \{0, 1\}$ for all i. AQ_1 is isomorphic to the complete graph K_2 where one vertex is labeled by the digit 0 and the other by 1. For $n \geq 2$, AQ_n is defined recursively by using two copies of $(n-1)$-dimensional augmented cubes with edges between them. We first add the digit 0 to the beginning of the binary strings of all vertices in one copy of AQ_{n-1}, which will be denoted by AQ^0_{n-1}, and add the digit 1 to the beginning of all the vertices of the second copy, which will be denoted by AQ^1_{n-1}. We now describe the edges between these two copies. Let $u = 0u_1u_2 \cdots u_{n-1}$ and $v = 1v_1v_2 \cdots v_{n-1}$ be vertices in AQ^0_{n-1} and AQ^1_{n-1}, respectively. Then u and v are adjacent if and only if one of the following conditions holds:

(1) $u_i = v_i$ for every $i \geq 1$. In this case, we call the edge (u, v) a *cross edge* and say $u = v^x$ and $v = u^x$.
(2) $u_i \neq v_i$ for every $i \geq 1$. In this case, we call (u, v) a *complement edge* and denote $u = v^c$ and $v = u^c$.

Throughout this paper, we denote the set of cross edges and complement edges in AQ_n by X_n and C_n respectively. Clearly, AQ_n is $(2n-1)$-regular, $|C_n| = |X_n| = 2^{n-1}$ and the edges in C_n (X_n) are independent. It is well-known that AQ_n is vertex-transitive. Another important fact is that the connectivity of AQ_n is $2n-1$ for $n \geq 4$. Some recent papers on augmented cubes include [3,6,13–15,17,21,22]. A few examples of augmented cubes are shown in Fig. 1.1.

2. Preliminaries

Our objective is to show that $\mathrm{smp}(AQ_n) = 2n - 1$, which is the best possible result, and that all optimal solutions are trivial. In this section, we present some results that will be useful in our quest. Since the strong matching preclusion problem is a generalization of the matching preclusion problem and the latter problem has been solved for AQ_n, we state the corresponding result.

Theorem 2.1 (*[6]*). *Suppose $n \geq 4$. Then $mp(AQ_n) = 2n - 1$. Moreover, every optimal matching preclusion set is trivial.*

Given that a Hamilton cycle in an even graph induces two edge-disjoint perfect matchings, the following result uses the "fault Hamiltonian" property as a sufficient condition in determining the strong matching preclusion number.

Proposition 2.2 (*[2]*). *Let G be an r-regular even graph with the property that $G - F$ is Hamiltonian for every $F \subseteq V(G) \cup E(G)$ where $|F| \leq r - 2$. Then $smp(G) = mp(G) = r$.*

However, we are unaware of any relationship between such "fault Hamiltonian" property and the classification of optimal strong matching preclusion sets. In order to apply Proposition 2.2, we need Hamiltonian results for AQ_n. Fortunately, such a result is known.