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On the complexity of injective colorings and its generalizations[☆]

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ABSTRACT

In this paper, we consider the complexity and algorithms of the injective coloring problem. We prove that it is NP-complete to determine the injective chromatic number even restricted to some special bipartite graphs. Furthermore, we show that for every $\epsilon > 0$, it is impossible to efficiently approximate the injective chromatic number of any bipartite graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless ZPP $=$ NP. For the max-injective coloring problem, we prove that there is a constant approximation algorithm on power chordal graphs with bounded injective chromatic number. We also devise a constant approximation algorithm for max-injective coloring some bipartite graphs. For the on-line injective coloring problem, we prove that First Fit (FF) injectively colors P_3-free graphs perfectly, where First Fit is an on-line algorithm that simply assigns the smallest available color to each vertex. We also prove that the number of colors used by *FF** for bipartite graph G is bounded by $\frac{3}{2}$ times the on-line injective chromatic number, where FF* is an on-line algorithm equivalent to FF proper coloring the complement of G. Moreover, we present an improved algorithm BFF, and prove that it is optimal for on-line injectively coloring bipartite graphs.

1. Introduction

An *injective k-coloring* of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any two distinct vertices u and v in $V(G)$ that have a common neighbor. The *injective chromatic number*, $\chi_i(G)$, of a graph G is the least k such that G admits an injective k-coloring. This concept was originated from complexity theory on Random Access Machines, and can be applied in the theory of error-correcting codes. In [6], Hahn et al. introduced these concepts, and they proved that $\Delta(G) \leq \chi_i(G) \leq \Delta(G)(\Delta(G) - 1) + 1$ (where $\Delta(G)$ denotes the maximum degree of G). They also characterized the extremal graphs. It is NP-complete to determine the injective chromatic number of graphs [6], and it is still NP-complete to determine its injective chromatic number for chordal graphs [7].

Some difficult combinatorial problems that are NP-complete in general admit polynomial time solutions by forbidding a kind of subgraphs (see [12,13] for example), or in bipartite graphs (including graph coloring). In [7], Hell et al. showed

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that the injective chromatic number of a tree can be computed efficiently. However, we show that it is still NP-complete to determine the injective chromatic number even restricted on bipartite graphs with some special properties. Furthermore, we show that for every $\epsilon > 0$, it is impossible to efficiently approximate the injective chromatic number of any bipartite graph within a factor of $n^{\frac{1}{3}-\epsilon}$ if $ZPP \neq NP$.

Let G be a graph with a weight function $\omega : V \rightarrow \mathbf{N}$. The *max-coloring problem* seeks to find a partition of the vertices of G into independent sets that minimizes the sum of the weights of the heaviest vertices in each independent set (this may be referred to as a weighted version of coloring problem). Pemmaraju, Raman and Varadarajan [14] showed that the max-coloring problem is NP-hard even restricted on interval graphs, and devised a simple 2-approximation algorithm for max-coloring interval graphs. Guan and Zhu [5] showed that the max-coloring problem can be solved in polynomial time on graphs of bounded path-width. Motivated by the allocating buffer application and digital signal processing applications, Govindarajan and Rengarajan [4] experimentally evaluate a first-fit strategy which produces a solution of max-coloring problem on circular-arc graph, with weight no more than 102.1% of the optimal weight.

In this paper we extend max-coloring problem to *max-injective coloring problem*. It is a combination of max-coloring and injective coloring. Given a graph G with a weight function $\omega : V \rightarrow \mathbf{N}$, the *max-injective coloring problem* is to find an injective vertex coloring $C_1, C_2, \ldots, C_k$ of G that minimizes $\sum_{i=1}^{k} max_{v\in C_i}\omega(v)$. Usually, $\sum_{i=1}^{k} max_{v\in C_i}\omega(v)$ is referred to as the weight of the injective coloring. When $\omega(v) = 1$ for all $v \in V(G)$, min $\sum_{i=1}^{k} max_{v\in C_i}\omega(v)$ is simply $\chi_i(G)$.

We prove that there is a constant approximation algorithm for solving the max-injective coloring problem on power chordal graphs with bounded injective chromatic number, and we devise a constant approximation algorithm for max-injective coloring some special bipartite graphs.

Motivated by the investigation of on-line proper coloring problem, we also study the *on-line version* of injective coloring. The injective coloring problem gets more complicated in the on-line situation. In this case, vertices of a graph are presented one at a time, and the algorithm has to assign a color irrevocably to a vertex as it comes in. The procedure depends only on the knowledge of the subgraph that has been revealed so far. To be precise, an *on-line injective coloring* of G is an algorithm that injectively colors G by receiving its vertices in some order $v_1, v_2, \ldots, v_n$, where the color of v_i is assigned depending only on the subgraph of G induced by $\{v_1, v_2, \ldots, v_i\}$.

After this slight change the problem becomes more complex. Actually, we find that the worst-case performance ratio between on-line and off-line injective coloring of a path on $2n$ vertices is at least $\frac{n}{2}$. Gyárfás and Lehel [3] introduced the concept of on-line coloring while translating a rectangle packing problem in dynamical storage allocation into coloring problem. Since then, on-line coloring of graphs has been investigated extensively. The optimal competitive ratio of on-line coloring problem is only slightly sublinear in general [10]. However, a constant competitive ratio is possible on interval graphs [3,9], and a logarithmic ratio can be achieved on bipartite graphs and sparse graphs [8]. Broersma, Capponi and Paulusma [1] proved that there exists an on-line competitive algorithm for the class of P_6-free bipartite graphs and P_7-free bipartite graphs, although P_6-free bipartite graphs do not admit a competitive algorithm [3].

In this paper, we prove that *First Fit* (FF) [3] perfectly injectively colors P_3-free graphs. The number of colors used by FF* on a bipartite graph G is bounded by $\frac{3}{2}$ of its on-line injective chromatic number, where FF* is an on-line algorithm equivalent to proper coloring the complement of G by FF. Moreover, we present an improved algorithm BFF, and prove that it is optimal for on-line injectively coloring bipartite graphs.

We use $\alpha(G)$ to denote *independent number* of G, that is the size of a maximum independent set in G. Let $AOL(G)$ be the set of all on-line injective coloring algorithms for a graph G, and let $\Pi(G)$ denote all the permutations of the vertices of G. For $A \in AOL(G)$ and $\pi \in \Pi(G)$, we use $\chi_i^A(G, \pi)$ to denote the number of colors used by A when the vertices of G are presented in the order of π. Let $\chi_i^A(G) = max_{\pi\in\Pi(G)}\chi_i^A(G, \pi)$, and refer to $\chi_i^A(G)$ as the *A-injective chromatic number* of G. Actually, $\chi_i^A(G)$ measures the worst case behavior of A on G.

The *on-line injective chromatic number* of G, denoted by $\chi_i^{OL}(G)$, is the smallest number of colors used by an algorithm in $AOL(G)$. Thus, $\chi_i^{OL}(G) = min_{A\in AOL(G)}\chi_i^A(G)$. OL can be considered to be an optimal on-line algorithm for injective coloring G. Obviously, $\chi_i^{OL}(G) \geq \chi_i(G)$.

Let $\mathscr{G}$ be a family of graphs. If $A \in AOL(G)$ for every graph $G \in \mathscr{G}$, we say that A is *an on-line injective coloring algorithm* for $\mathscr{G}$ and write $A \in AOL(\mathscr{G})$. Let $f(x)$ be a function on a single variable x. An algorithm $A \in AOL(\mathscr{G})$ is said to be *f-competitive* for $\mathscr{G}$ if $\chi_i^A(G) \leq f(\chi_i(G))$ for every $G \in \mathscr{G}$, i.e., the number of colors used by A on G is bounded from above by a function depending only on the injective chromatic number of G. When $f(x) = cx$ for some constant c, an f-competitive algorithm is called *c-competitive*, and c is referred to as the *competitive ratio*.

In fact, there are many graphs which have no competitive algorithms. Take P_{2n} for example. Any algorithm A uses at least n colors when the vertices of P_{2n} are revealed under an order π where a maximum independent set I of G emerges first, without knowledge of coming vertices. Compared with 2 (the injective chromatic number of P_{2n}), A is not competitive for P_{2n}. This negative results have led to the definition of a weaker form of competitiveness. An algorithm $A \in AOL(\mathscr{G})$ is said to be *on-line c-competitive* if there exists a constant c such that $\chi_i^A(G) \leq c\chi_i^{OL}(G)$ for every $G \in \mathscr{G}$, and c is referred to as the *on-line competitive ratio*.

The paper is organized as follows. In Section 2, we prove the NP-hardness of determining the injective chromatic number of some special bipartite graphs. Furthermore, we show that for every $\epsilon > 0$, it is impossible to efficiently approximate the injective chromatic number of any bipartite graph within a factor of $n^{\frac{1}{3}-\epsilon}$ if $ZPP \neq NP$. Then, we prove that there is a constant