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# Semantics and proof-theory of depth bounded Boolean logics[☆]

Marcello D'Agostino [a,*], Marcelo Finger [b], Dov Gabbay [c,d,e]

[a] *Department of Economics and Management, University of Ferrara, Italy*
[b] *Department of Computer Science, University of Sao Paulo, Brazil*
[c] *Bar Ilan University, Israel*
[d] *King's College London, United Kingdom*
[e] *University of Luxembourg, Luxembourg*

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ABSTRACT

We present a unifying semantical and proof-theoretical framework for investigating depth-bounded approximations to Boolean Logic, namely approximations in which the number of nested applications of a single structural rule, representing the classical Principle of Bivalence, is bounded above by a fixed natural number. These approximations provide a hierarchy of tractable logical systems that indefinitely converge to classical propositional logic. The framework we present here brings to light a general approach to logical inference that is quite different from the standard Gentzen-style approaches, while preserving some of their nice proof-theoretical properties, and is common to several proof systems and algorithms, such as KE, KI and Stålmarck's method.



## 1. Introduction

Logic is informationally trivial and, yet, computationally hard. This is one of the most baffling paradoxes arising from the traditional account of logical consequence. Triviality stems from the widely accepted characterization of deductive inference as "tautological", in the sense of being *uninformative* or *non-ampliative*: the information carried by the conclusion is (in some sense) contained in the information carried by the premisses. Computational hardness stems from the well-known results showing that most interesting logics are either undecidable or likely to be intractable. In classical quantification theory, the tension between the alleged triviality of logical consequence and the established fact that it admits of no mechanical decision procedure was described by Jaakko Hintikka as a true "scandal of deduction" [38].[1] In Hintikka's view, the scandal was confined to first-order logic: by virtue of its decidability, Boolean Logic was prima facie consistent with the traditional view that logical consequence is uninformative. However, Cook's theorem [14] – according to which Boolean logic is tractable if and only if $\mathscr{P} = \mathscr{NP}$ – strongly suggests that even in the innocuous-looking domain of Boolean languages, logical consequence is probably far from being uninformative.[2] This unsettling situation is also related to the so-called *problem of logical omniscience*: if logic were informationally trivial, then a rational agent should *always* be aware that a certain sentence

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* Corresponding author.
*E-mail addresses:* dgm@unife.it (M. D'Agostino), mfinger@ime.usp.br (M. Finger), dov.gabbay@kcl.ac.uk (D. Gabbay).

1 For a recent criticism of Hintikka's proposed solution, see [47].

2 On this point see [23], where some of the ideas underlying the present work were first put forward, and [21].

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is a logical consequence of the data. However, this is clearly not the case when the logic in question lacks a feasible decision procedure.

A typical response to this paradox is that logical systems are *idealizations* and, as such, not intended to faithfully describe the actual deductive behavior of rational agents. As Gabbay and Woods put it:

> A logic is an idealization of certain sorts of real-life phenomena. By their very nature, idealizations misdescribe the behavior of actual agents. This is to be tolerated when two conditions are met. One is that the actual behavior of actual agents can defensibly be made out to approximate to the behavior of the ideal agents of the logician's idealization. The other is the idealization's facilitation of the logician's discovery and demonstration of deep laws. [34, p. 158]

This raises what can be called the *approximation problem* that, in the context of logical systems, can be concisely stated as follows:

**Approximation Problem.** *Can we define, in a natural way, a hierarchy of logical systems that indefinitely approximate a given idealized Logic (e.g., Boolean Logic) in such a way that, in all practical contexts, suitable approximations can be taken to model the inferential power of actual, resource-bounded agents?*

Stable solutions to this problem are likely to have a significant practical impact in all research areas – from knowledge engineering to economics – where there is an urgent need for more realistic models of deduction, and involve an imaginative re-examination of logical systems as they are usually presented in the literature.[3]

The idea of approximating Boolean logic via tractable subsystems of increasing inferential power has received some attention in Computer Science and Artificial Intelligence [12,25,26,15,48,29,30,32,33,31], but comparatively little attention has been devoted to embedding such efforts in a systematic proof-theoretical and semantic framework. In this paper we aim to fill this gap and propose a unifying approach. Because of the nature of this task, some of the ideas and results presented here are new, while (variants of) others have repeatedly and independently come up in the logical literature – sometimes within different communities that hardly communicate with each other – in such a way that is quite difficult to trace their origin or give proper credit for each of them (see Section 4 for an attempt in this direction). We hope that the present contribution may also shed light on these ideas, clarify their connections, and help building bridges between different research areas.

The unifying approach presented in this paper is rooted in a long-standing research programme started in [41,42,44] and further developed in [16,24,45,18,29–31,23,20]. The heuristic hard-core of this research programme is concisely described by the following statements:

- The classical meaning of the Boolean operators is *overdetermined* by the standard truth-functional semantics and this fact is probably responsible for the lack of a suitable inferential semantics[4] for them.
- Boolean inferences are better construed as arising from the interplay between a weaker basic semantics for the logical operators and the purely structural principles of *Bivalence* and *Non-Contradiction*.
- A suitable inferential semantics for the standard classical operators is given by the *introduction* rules of the system KI [41,16,45] combined with the *elimination* rules of the system KE [42,16,18,19]; these rules are not complete for Boolean Logic, but completeness is achieved by adding suitable structural rules corresponding to the two principles of Bivalence and Non-Contradiction.
- The separation between the inferential role played by the meaning of the logical operators and the inferential role played by the structural principles naturally prompts for the definition of *depth-bounded* approximations in which nested applications of the structural rule expressing the Principle of Bivalence are limited above by a fixed natural number.

A similar heuristic was implicit in the work of Gunnar Stålmarck [49] and has been, independently, pursued [48,8–10] with more practical motivations, leading to efficient and widely used techniques for software verification (see Section 4 for further details).

The present paper is a systematic development of ideas put forward in [23]. We first show (Section 2) that the set of rules obtained by generalizing the introduction rules of KI and the elimination rules of KE to a language with *arbitrary* Boolean operators defines a tractable "natural deduction" system for the chosen operators that (i) is a logic in Tarski's sense, (ii) enjoys the subformula property (with no increase in proof-length) and (iii) allows for a semantic characterization in terms of non-compositional partial valuations or, equivalently, of a certain kind of non-deterministic matrices. Next (Section 3) we investigate the hierarchy of tractable extensions of this basic system that are defined by bounding the application of the Principle of Bivalence in a variety of ways.

[3] This kind of approximation problem arises from the computational idealizations typically made by logical models. But clearly these models involve other kinds of idealizations (see, for instance [28] on this point) that define other kinds of approximation problems.

[4] By "inferential semantics" we mean the approach that identifies the meaning of the logical operators with the role they play in basic inferences. This is usually related to the meaning-as-use approach advocated by the later Wittgenstein and to Gentzen's suggestion in [35] that the rules of his system of Natural Deduction could be taken as definitions of the logical operators. (Indeed, he proposed that the introduction rules would be sufficient for this purpose and that the elimination rules could be ultimately "justified" in terms of the introduction rules.)