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A direct and local method for computing polynomial Pythagorean-normal patches with global G^1 continuity

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ABSTRACT

We present a direct and local construction for polynomial G^1 spline surfaces with a piece-wise Pythagorean normal (PN) vector field. A key advantage of our method is that the constructed splines possess exact piece-wise rational offsets without any need for reparametrisations, which in turn means that no trimming procedure in the parameter domain is necessary. The spline surface consists of locally constructed triangular PN macro-elements, each of which is completely local and capable of matching boundary data consisting of three points with associated normal vectors. The collection of the macro-elements forms a G^1-continuous spline surface. The designed method is demonstrated on several examples.

1. Introduction

Curves and surfaces satisfying a certain *Pythagorean* property of their tangent or normal vector fields have become an intensive research topic in recent years. Investigating their properties and applications significantly influenced research in related theoretical as well as applied disciplines, and nowadays one can find a large number of papers and other contributions related to this interesting concept [1,2].

This paper is devoted to surfaces in 3-space whose normal vectors satisfy the Pythagorean property, the so-called PN surfaces. Rational PN surfaces were defined in [3] as a surface counterpart of Pythagorean hodograph (PH) curves [4]. It holds that PH curves in the plane and PN surfaces in 3-space share some common properties, for instance they both yield rational offsets. This property is highly appreciated in technical practice since for general free-form NURBS curves and surfaces an exact (piece-wise) rational parametric representation of their offsets is not available, and approximate techniques for computing and interrogating their offsets are thus needed.

Nonetheless, when considering only the rationality of offsets as a main feature of PH curves or PN surfaces then other useful properties might be overlooked. In the curve case, another very important practical application is based on the fact that the parametric speed (or the length element), and thus also the arc length, of polynomial PH curves is also polynomial. This is important, for instance, when formulating efficient real time interpolator algorithms for CNC machines. The area element and the surface area are then the analogues in the surface case: they are both polynomial for polynomial PN surfaces. This feature is useful for instance in CNC painting. This shows the prominent role of polynomial PH curves and PN surfaces within their rational families.

Despite the fact that both PH curves and PN surfaces belong among hypersurfaces with a Pythagorean property, one can find important differences between these two classes. For instance, PH curves were first introduced as planar *polynomial* shapes, including a compact formula for their description based on Pythagorean polynomial triples, whereas a description of *rational* PN surfaces using their duals was first revealed in [3]. This has clear consequences for formulating interpolation/approximation algorithms with these shapes. There exist many Hermite interpolation results for polynomial PH curves [1,2], but there are not many algorithms for PN surface interpolation. Moreover, only select few are *direct* PN surface algorithms and the majority of those use rational PN surfaces. A direct PN algorithm is a construction of the object together with its PN parametrisation (i.e., no reparametrisation is required).

In contrast, results of *indirect* PN algorithms are surfaces which become PN only after a suitable rational reparametrisation, i.e., one does not obtain a polynomial PN surface but a rational one. For instance, in [5] a method for the construction of exact offsets of quadratic triangular Bézier surface patches was designed. These patches are in fact PN surfaces but their PN parametrisations were obtained only via a certain reparametrisation. A nice approach also

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based on reparametrisations was formulated in [6], using surfaces with linear normals [7].

As for direct methods, a scheme with triangular patches on parabolic Dupin cyclides was designed in [8], interpolation of triangular data using the support function was studied in [9], and using bicubic Coons patches in the isotropic model for the construction of smooth PN surfaces was investigated in [10].

The key advantage of direct PN interpolation techniques is obvious: as no reparametrisation is required, one does not need to apply trimming in parameter space. Nevertheless, as in the case of indirect approaches, all above-mentioned direct methods yield rational PN surfaces, and thus cannot be used when polynomial parametrisations are required. Only recently, the first method solving the Hermite problem directly, and thus yielding polynomial PN parametrisations, was formulated in [11]. However, the method is global and requires solving a global linear system; the locality of e.g. the (rational) method presented in [10] is lost.

In the present paper, based on reformulating the approaches taken in [6] and [11], we solve the challenging problem of designing a PN Hermite interpolation method which

- is local, i.e., a PN macro-element is computed only from vertex and normal data of one triangle at a time;
- is direct, i.e., it yields polynomial PN macro-elements with no need for reparametrisations;
- yields globally G^1-continuous PN spline surfaces.

We describe our algorithm in Section 3, present examples in Section 4, and conclude the paper in Section 5. But before all that, we recall some preliminary notions and set notation in the following section.

2. Preliminaries

In this section we recall some fundamental facts about surfaces with rational offsets and rational curves on them.

2.1. PN surfaces and PSN curves

For the sake of completeness, we first recall the definition of PN surfaces.

Definition 2.1. Let $\mathcal{X}$ be a rational surface for which there exists a parametrisation $\mathbf{x}(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying the condition

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = \sigma^2, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm, $\sigma(u, v)$ is a rational function, and $\mathbf{x}_u$ and $\mathbf{x}_v$ are partial derivatives of $\mathbf{x}$ with respect to u and v, respectively. Then $\mathcal{X}$ is called a *surface with a Pythagorean normal vector field* (or a *PN surface*) and condition (1) is referred to as *PN condition* or *PN property*. A parametrisation satisfying the PN condition is called a *PN parametrisation*. If every parametrisation of $\mathcal{X}$ is PN, we call $\mathcal{X}$ a *proper* PN surface. If there exist both PN and non-PN parametrisations of $\mathcal{X}$ then we speak about a *non-proper* PN surface.

A distinguishing property of PN surfaces is that they admit two-sided rational δ-*offset surfaces*

$$\mathbf{x}_\delta = \mathbf{x} \pm \delta \frac{\mathbf{N}}{\|\mathbf{N}\|} = \mathbf{x} \pm \delta \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sigma}, \tag{2}$$

where $\mathbf{x}(u, v)$ is a PN parametrisation of $\mathcal{X}$ and $\mathbf{N}(u, v)$ is a normal vector (at regular points of $\mathcal{X}$).

Moreover, as it holds

$$\begin{vmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_u \cdot \mathbf{x}_v & \mathbf{x}_v \cdot \mathbf{x}_v \end{vmatrix} = EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2 \tag{3}$$

with E, F, G the coefficients of the first fundamental form, and the squared *area element* has the form

$$dA^2 = (EG - F^2)\, du^2 dv^2, \tag{4}$$

then PN surfaces are simultaneously *surfaces with a rational area element* in $\mathbb{R}^3$. In addition, all polynomial PN surfaces (with polynomial area element) possess piece-wise polynomial surface area

$$A(u, v) = \iint \sqrt{EG - F^2}\, du dv = \iint |\sigma|\, du dv.$$

Let $\mathcal{X}$ be a rational surface and $\mathcal{C} \subset \mathcal{X}$ be a rational curve on it given by the parametrisation $\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$ for some rational functions $u(t)$ an $v(t)$. The normal vector field of the surface along $\mathcal{C}$ is expressed as

$$\mathbf{N}(u(t), v(t)) = \mathbf{x}_u(u(t), v(t)) \times \mathbf{x}_v(u(t), v(t)). \tag{5}$$

The δ-*offset of the given surface along its curve* is then defined by

$$\mathbf{x}(u(t), v(t)) \pm \delta \frac{\mathbf{N}(u(t), v(t))}{\|\mathbf{N}(u(t), v(t))\|}. \tag{6}$$

Of course, the curve (6) is not rational, in general. Indeed, the formula gives a rational mapping if and only if there exists a rational function $\sigma(t)$ such that the following (Pythagorean) condition is satisfied:

$$[\mathbf{x}_u(u(t), v(t)) \times \mathbf{x}_v(u(t), v(t))]^2 = \sigma^2(u(t), v(t)). \tag{7}$$

Then we say that the parametrisation $\mathbf{x}(u(t), v(t))$ satisfying (7) *admits Pythagorean surface normals with respect to* $\mathcal{X}$, and is shortly called a *PSN parametrisation*. A curve $\mathcal{C} \subset \mathcal{X}$ admitting a PSN parametrisation is called a *PSN curve*; see [12].

The PSN condition (7) can be satisfied for some curves despite the fact that the PN condition (1) does not hold for the given surface parametrisation. On the other hand, when the parametrisation $\mathbf{x}(u, v)$ of the surface $\mathcal{X}$ is PN, then any parametrisation $\mathbf{x}(u(t), v(t))$ of the curve $\mathcal{C} \subset \mathcal{X}$ is PSN. Nevertheless, we emphasise that not every rational curve on a PN surface is PSN; this can happen when the surface is a non-proper PN surface.

2.2. Polynomial PN triangles

Our goal is to construct a smooth piece-wise polynomial PN surface interpolating given G^1 data, i.e., points and normals. We assume that the input data are organised in a triangular manifold mesh (with or without boundary). Before proceeding, we first reformulate the expressions involved in the PN property for Bézier triangular patches.

With $\mathbf{i} = (i, j, k)$, $|\mathbf{i}| = i+j+k$ and $i, j, k \geq 0$, a triangular Bézier surface patch of degree n is defined as

$$\mathbf{x}(\mathbf{u}) = \sum_{|\mathbf{i}|=n} \frac{n!}{i!j!k!} \mathbf{x}_{\mathbf{i}} u^i v^j w^k \tag{8}$$

with barycentric coordinates $\mathbf{u} = (u, v, w)$, $u, v, w \geq 0$, $u + v + w = 1$, and control points $\mathbf{x}_{\mathbf{i}} \in \mathbb{R}^3$; see [13]. The domain of the patch is a triangle $\triangle \subset \mathbb{R}^2$.

The first directional derivatives with directions parallel to the edges of $\triangle$ are

$$\begin{aligned} \mathbf{x}_u(\mathbf{u}) &= n \sum_{|\mathbf{i}|=n-1} \Delta_u \mathbf{x}_{\mathbf{i}} u^i v^j w^k, \\ \mathbf{x}_v(\mathbf{u}) &= n \sum_{|\mathbf{i}|=n-1} \Delta_v \mathbf{x}_{\mathbf{i}} u^i v^j w^k, \\ \mathbf{x}_w(\mathbf{u}) &= n \sum_{|\mathbf{i}|=n-1} \Delta_w \mathbf{x}_{\mathbf{i}} u^i v^j w^k, \end{aligned} \tag{9}$$