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# Refinable bi-quartics for design and analysis

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ABSTRACT

To be directly useful both for shape design and a thin shell analysis, a surface representation has to satisfy three properties: (1) be compatible with CAD surface representations, (2) yield generically a good highlight distribution, and (3) offer a refinable space of functions on the surface. Here we propose a new construction, based on a number of recently-developed techniques, that satisfies all three criteria. The construction converts quad meshes with irregularities, where more or fewer than four quads meet, to $C^1$ (or, at the cost of more pieces, $C^2$) bi-4 surface consisting of subdivision rings for the main body completed by a tiny $G^1$ cap.



## 1. Introduction

Modern piecewise polynomial constructions are based on careful parameterizations to achieve generically good highlight line distributions also where more or fewer than four 4-sided pieces meet. This representation is directly compatible with current CAD surface standards. Remarkably, such $G^k$ constructions automatically provide a $C^k$ space of functions on the surfaces [1,2]. However, nested refinement of the $G^k$ representation requires careful tracking of the original $G^k$ edges to ensure that a solution obtained at one level of refinement is not lost at the next finer one. Already when $k = 1$, the characterization of all possible degrees of freedom, even when joining just one pair of patches, is not easy [3]. For specific high-end surface constructions, where the degrees of freedom under refinement have been explicitly characterized, they are heterogeneously distributed [4].

An alternative is to model multi-sided surface pieces with an infinite sequence of nested polynomial surface rings so that nested refinement is built-in. Such generalized subdivision is exemplified by Catmull–Clark (CC) subdivision [5]. Unfortunately many parts of the CAD pipeline are not set up for infinite recursive definition. More importantly, as Fig. 1 illustrates, CC subdivision produces poor highlight lines even for simple configurations, such as joining two crossing pipes. Note the characteristic 'pinching' of highlight lines near the 6-valent irregularity.

Retaining the best of subdivision and geometrically continuous surface constructions leads to the approach of this paper. The key observation is that in practice, analysis works with a maximal anticipated refinement level $\ell$ for a given geometric design. We therefore propose to model multi-sided surface pieces as a sequence of $C^1$ surface rings closed off by an $n$-sided $G^1$-cap at the final anticipated refinement level.

a. Since the final surface consists of a fixed number of $3n\ell + 4n$ bi-4 (bi-quartic) surface pieces, it is CAD representable.
b. Since the surface rings and final cap carefully follow a guiding shape, good highlight line distribution as in Fig. 1 is observed, without exception for an obstacle course of challenging configurations.
c. Since the maximal refinement level at the irregularity is realized by the $G^1$-cap, it need not be refined. And since the sequence of surface rings forms a $C^1$ complex, nested refinability amounts to standard regular spline refinement.

While non-trivial in its derivation, the bi-4 surface is efficiently constructed via pre-tabulated operators, akin to the more light-weight subdivision stencils of CC subdivision. The number of surface rings can be varied to suit the application.

*Structure of the paper.* Section 2 explains the input and basic tools used for the constructions: the corner jet constructor, maps of total degree and characteristic parameterizations. Section 3 describes the guide and Section 4 a guided subdivision. Section 6 characterizes the eigenstructure of this subdivision and reveals how it is inherited from a guide surface. Section 5 reformulates the bi-4 guided subdivision to look more like traditional CC subdivision. Section 7 sketches how $G^1$ caps yield a high-quality, hybrid construction with finitely many patches. Section 8 illustrates the surface quality according to alternatives and trade-offs that lead to the preferred bi-4 construction presented here.

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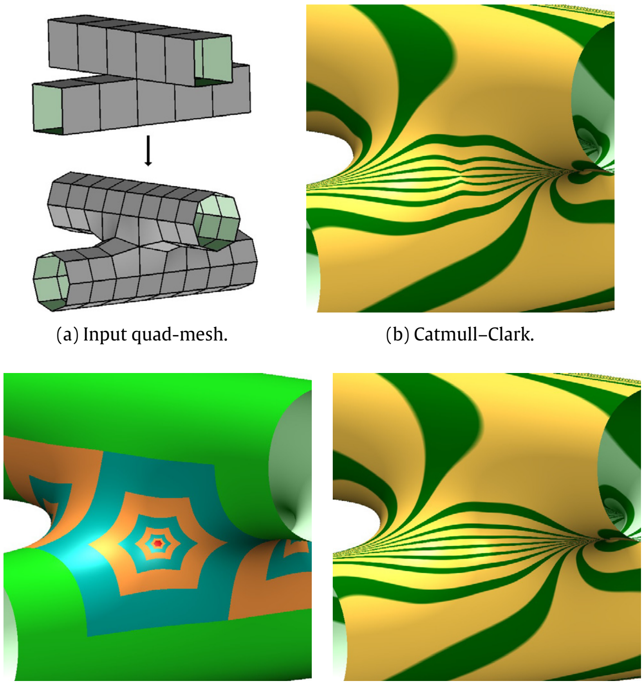
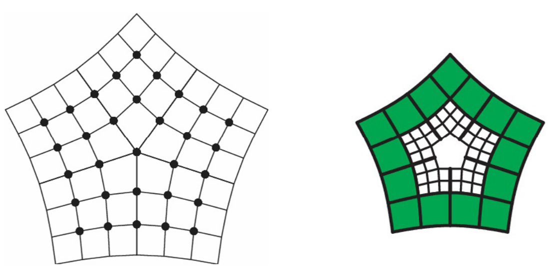


(a) Input quad-mesh. (b) Catmull–Clark. (c) Bi-4 guided rings + cap. (d) New finite bi-4.

**Fig. 1.** (a) Input mesh with (*top*) adjacent nodes of valence 6 and (*bottom*) once refined. (b) Pinched highlight lines typical of Catmull–Clark subdivision surfaces. (c) Regular surface of degree bi-3 (green) and $C^1$ bi-4 guided rings with tiny $G^1$ bi-4 cap (red) covering the 6-sided region with a guided surface with (d) good highlight line distribution. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) **c**-net extended by 1 layer. (b) Bi-3 ring + tensor-border.

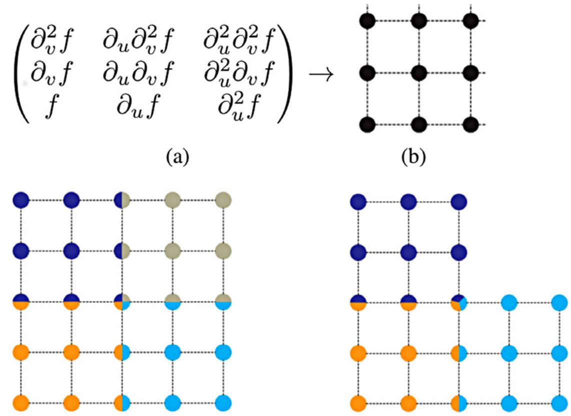
**Fig. 2.** B-spline-like irregular control net and tensor-border. (a) Extended **c**-net for $n = 5$. (b) Schema of surface ring (green) and its tensor-border (of degree 3 and depth 2 = inner grid of BB-coefficients). The tensor-border is the input for the multi-sided construction. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

## 2. Definitions and setup

This section characterizes the input, operators, $C^1$ polynomial caps of total degree, and the reparameterizations used to define first the guide and then the final surface.

### 2.1. A B-spline-like control net for irregular layout

We consider as input a network of quadrilateral facets, short quads. Nodes where four quads meet are regular, otherwise *irregular*. We assume that each irregular node is surrounded by at least one layer of regular nodes. Fig. 2a shows the **c**-net (bullets) of an isolated node of valence $n = 5$. The **c**-net consists of the irregular node plus $6n$ nodes forming two layers of quads surrounding




(a) (b) (c) (d)

**Fig. 3.** Corner jet constructor $[f]^d_{3\times 3}$ at work. (a) Hermite data as partial derivatives converted to (b) BB-form assembled, by averaging $3 \times 3$ jets, into (c) a patch of degree bi-4. (d) *L*-shaped sector of the tensor-border **t**. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

it. Typically a third layer is added for evaluation of local shape. The extra layer provides a surrounding surface (green in Fig. 2b). This allows tracing the highlight line distribution [6] across the transition where quality is as important and challenging as the internal quality of the cap.

Each $4 \times 4$ sub-grid of nodes is interpreted as the B-spline control points of a bicubic tensor-product spline surface. Except at the irregular node, well-known formulas can be applied to convert the B-spline form to Bernstein–Bézier form (see e.g. [7,8]). The tensor-product Bernstein–Bézier (BB) form of bi-degree $d$ is

$$\mathbf{p}(u,v) := \sum_{i=0}^{d}\sum_{j=0}^{d} \mathbf{p}_{ij} B_i^d(u) B_j^d(v)\,,$$

$$(u,v) \in \square := [0..1]^2, \quad \text{where } B_k^d(t) := \binom{d}{k}(1-t)^{d-k}t^k$$

are the BB-polynomials of degree $d$ and $\mathbf{p}_{ij}$ are the BB coefficients. Fig. 2b also shows the $C^2$ prolongation of this surface ring, i.e. Hermite data represented as a grid (black) of bi-3 BB-coefficients. Specifically, the BB-coefficients $\mathbf{p}_{ij}$, $i = 0,\ldots,3$, $j = 0,\ldots,2$, represent Hermite data of order 2 along one boundary curve $v = 0$. Degree-raised to bi-degree 4, we call these data $\mathbf{t}_{CC}$. In the remainder of this paper, we refer to second-order Hermite data of degree 4 along the loop of boundary curves as one of

$\mathbf{t}, \mathbf{h}$ = a tensor-border of degree 4 and depth 2.

We will construct tensor-product patches and tensor-borders with the help of jet constructors

$[f]^d_{i\times j}$, the corner jet constructor,

expresses, at a corner of the domain square $[0..1]^2$, the expansion of a function $f$ of order $i-1$ in $u$ and $j-1$ in $v$ in BB-form of bi-degree $d$. That is, $[f]^d_{i\times j}$ outputs $i \times j$ BB-coefficients (see Fig. 3a,b for $i = 3 = j$). Fig. 3c displays four corner jet constructors $[f]^4_{3\times 3}$ merged to form a bi-4 patch by averaging the overlapping BB-coefficients.

Fig. 3d illustrates the analogous assembly of an *L*-shaped sector of the tensor-border by applying and averaging a jet constructor at three corners.

Several steps of the surface construction use a simple symmetric rule, called C2-rule in the following and illustrated in Fig. 4: two curve segments (of the same degree) in BB-form join