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Dimensional perturbation of rigidity and mobility

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ABSTRACT

Mechanisms, defined as assemblies of dimensioned rigid bodies linked by ideal joints, can be partitioned in three mobility states: the rigid state (where bodies can have only one position relative to each other), the mobile state (where bodies can move relatively to each other) and the impossible state (where bodies dimensions and specified joints cannot lead to a feasible assembly). It is also clear that although bodies dimensions can vary in a continuous way, assemblies may experience quite abrupt changes across those states. This paper proposes a new approach to this problem with the goal of being able to predict the mobility class of an assembly of arbitrary complexity, and how it can be affected by a perturbation of the dimensions of its bodies. It does so by proposing a simple and general state transition framework including the three above defined states and seven transitions describing how a dimensional perturbation can affect them. Using this framework, the mobility of a mechanism is easier to capture and predict, using only dimensional (u) and positional (p) parameters involved in an appropriate equation ($F(u, p) = 0$). This is achieved by focusing on how $F()$ behaves when u and p get perturbed, and the impact of this reaction on the mobility state of the assembly. As a result of this more mathematic approach to the problem, previously used notions of iso-constraint, over-constraint and paradoxical assembly, traditionally used to describe such assemblies, can be rigorously defined and thus clarified.

1. Introduction

We are interested in designing and manufacturing physical mechanisms and understanding their properties in a formal setting. Variations in design can be obtained by perturbing dimensions, but this has an effect on the resulting mobility of the mechanism. Therefore, starting from an initial state, the point is to converge to a final design with the same mobility. Furthermore, a physical mechanism can never be manufactured at nominal dimensions. The point is also to understand how dimensional uncertainty can influence the properties of factory-made mechanisms.

The goal of this paper is to simplify definitions and investigation of properties of mechanisms. The focus is put on rigidity and mobility, which are widely studied [1–5]. The concept of mechanism used in this theory is very general since only basic semantic is involved. Dimensional parameters noted u define the dimensions of rigid bodies. They are set by the designer as input values of the equation. Positional parameters noted p define relative positions of rigid bodies. They are the unknowns of the equation. Constraints, joints, links, etc. complete the system by mixing u and p parameters into an equation $F(u, p) = 0$. Only general properties of the equation are used. How rigid bodies and constraints are involved is not detailed, provided it results in a $F(u, p) = 0$ type equation where function F features the appropriate smoothness. The theory proposed in this paper is valid whether group theory, graph theory, Cartesian or non Cartesian, relative or absolute modeling [3,6–14] is used to set up the equation.

The behavioral principle is to deliberately ignore the formulation of the equation and to focus on how function F behaves under parameters perturbations. Given (u_0, p_0) a couple of dimensional and positional parameters defining a mechanical assembly, meaning that $F(u_0, p_0) = 0$, the question is to completely figure out what can happen when u_0 and/or p_0 is perturbed. Does the assembly still exist? If yes, is rigidity or mobility saved or lost?

To reach this goal, the first step is to define states of the assembly. An assembly can be either "rigid", "mobile" or "impossible". States definitions are inspired by considering the positional parameter under a fixed dimensional parameter. The second step is to define all transitions from one state to another, postulating that the cause of a transition is a dimensional perturbation. In other words, can an assembly switch from any state to any other state, is there

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any sink state? It is well known that mechanism design must cope with non generic situations [1,2,15–20]. This is the reason why the present paper thoroughly investigates the nature and the effect of dimensional perturbations.

The landscape being dressed through the states and transitions, sufficient conditions on function F are investigated in order to understand how differential properties are related to behaviors. Surprisingly, it is proven that some transitions are never possible due to smoothness and invariance properties of function F. Furthermore, it is proven that some transitions are not accessible under first order condition: second order conditions must be investigated. These concepts are compared to traditional definitions like iso-constrained, over-constrained, paradoxical, thus bringing precision and clarification. Each situation is illustrated by an actual mechanism: either a planar rigid bars arrangement, or a crankshaft.

Section 2 recalls the classical Kutzbach–Grubler mobility index, as it is mentioned all through the paper for comparison. Section 3 compares the present approach with combinatorial rigidity. Section 4 defines the notion of mechanism at the appropriate level of abstraction. Section 5 provides mathematical definitions of the states. Section 6 defines and illustrates all possible transitions from one state to another. The important result is related to transitions from the "impossible" state. Section 7 exhaustively investigates sufficient conditions for state and transitions based on first order differential properties of function F. It also provides a track for second order analysis through bifurcation theory. After the state transition approach is set, Section 8 provides comparison with traditional notions. Section 9 exemplifies the analysis on the very classical crankshaft mechanism. Depending on the nature of the crank–rod joint, this mechanism exhibits the most interesting features presented in the theoretical sections. Finally, Section 10 concludes just before the mathematical theorems and lemmas of the Appendix.

2. Computing mobility index

Existing formula to compute the mobility index (also named the number of degrees of freedom) of a mechanism is known as the Kutzbach–Grubler criterion, as thoroughly investigated in [2] and references therein. Consider a mechanism involving n_b bodies through n_j joints. Let n be the sum of degrees of freedom of joints, each one considered independently. Then, the Kutzbach–Grubler criterion computes the so called mobility index δ of the mechanism by

$$\delta = n - g(n_j - n_b + 1) \qquad (1)$$

where g is a constant number depending on the nature of the mechanism. In fact, g is the number of degrees of freedom of an unconstrained arbitrary body. Planar and spherical mechanisms require $g = 3$. 3D mechanisms require $g = 6$. The mobility index is advantageously computed by using a graph including n_b nodes, respectively associated with rigid bodies, and n_j arcs, respectively associated with joints. Each arc is labeled with the joint's degrees of freedom, meaning that n is the sum of all labels. This graph is illustrated each time in the following. It is proven in [2] that (1) is questionable in many practical situations, and the connection with the present work is interesting.

3. Combinatorial rigidity

The purpose of this section is to compare the present approach with the combinatorial rigidity theory [3–5]. The reader who is not familiar with this theory can skip this section. Combinatorial rigidity investigates structures made of rigid bars connected at their end points by spherical joints. The combinatorial aspect is the logical graph underlying the structure: bars are modeled by edges and joints are modeled by vertices. The theory provides criteria for generic rigidity of such structures.

The first drawback is that it handles spherical joints only, while many other kinds of joint are needed in mechanical design: cylindrical, prismatic, revolute, coplanar, etc. Mechanism of Figs. 6, 8 and 18 involve prismatic joints. Assembly of Figs. 10 and 14 could be modeled by a graph made of rigid bars connected at their end points, but this would artificially increase complexity and dimensioning.

The second drawback is that combinatorial rigidity does not explicitly deal with dimensional parameters. Variables of the edge function and its derivative, the rigidity matrix (the key feature of the theory) are vertices coordinates of the graph structure. The theory is not suited to investigate dimensional parameters influence. Indeed, combinatorial rigidity handles genericity from the vertices coordinates points of view, which makes a big difference with the present approach. For example, assemblies of Figs. 2 and 4 are generically rigid form the combinatorial theory point of view: any small perturbation of vertices (joints) coordinates yields an assembly of the same nature. From the mechanical designer point of view, the situation is totally different. Any small perturbation of bars lengths of assembly in Fig. 2 yields an assembly of the same nature. It is generic in this sense. Conversely, there exists arbitrary small perturbations of bars lengths of assembly in Fig. 4 that makes it impossible. It is not generic in this sense. Bars lengths are not independent, they must fit a special relationship, as investigated by Proposition 3. This makes a fundamental difference from the manufacturing point of view.

4. Abstract mechanism

The concept of abstract mechanism is set up to deal with any kind of mechanism, while saving the semantic of dimensional vs. positional parameters. It should be understood as an abstract way to model any mechanism. In the following, symbols U and P represent parameters spaces. For sake of generality, these spaces should be manifolds and used with local maps to finite dimensional euclidean spaces. Nevertheless, for simplicity and since local analysis is performed all through the paper, parameters spaces are defined as finite dimensional spaces. The only exception is Proposition 1 where the nature of spaces U and P is particularly considered.

Definition 1. An abstract mechanism is a four-tuple $\mathcal{M} = (U, P, E, F)$ where $U = \mathbb{R}^n$ is the space of dimensional parameters, $P = \mathbb{R}^m$ is the space of positional parameters, $E = \mathbb{R}^k$ and $F : U \times P \to E$ is a smooth function.

Function F captures the nature of the mechanism. It involves dimensional and positional parameters according to rigid bodies and joints. Function F is supposed to be twice continuously differentiable despite it is generally analytic. Furthermore, the derivative of F with respect to all its variables is supposed to be full rank. The equation of the mechanism is then $F(u, p) = 0$, which is a shortcut to deal with k scalar equations.

Definition 2. An assembly of the abstract mechanism $\mathcal{M}$ is a couple $(u, p) \in U \times P$ such that $F(u, p) = 0$. The set of all assemblies is noted

$$Z = \{(u, p) \in U \times P,\ F(u, p) = 0\}.$$

Definition 2 makes the connection with the classical notion of workspace. This concept, which is very popular in robotics [21], is used to describe the space of feasible positions of mobile