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The use of a particle method for the modelling of isotropic membrane stress for the form finding of shell structures

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HIGHLIGHTS

- Derivation of equilibrium equations for loaded shell structures with a varying isotropic stress.
- The application of the finite element method for the numerical implementation.
- The introduction of a new particle method for the numerical implementation.

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ABSTRACT

The best known isotropic membrane stress state is a soap film. However, if we allow the value of the isotropic stress to vary from point to point then the surface can carry gravity loads, either as a hanging form in tension, or as a masonry shell in compression. The paper describes the theory of isotropic membrane stress under gravity load and introduces a particle method for its numerical simulation for the form finding of shell structures.

1. Introduction

In conventional structural design the geometry of a structure is first chosen more or less arbitrarily and then analyzed to establish how well it performs. It is then modified to improve its performance and this cyclic process is continued until the designers are satisfied. Michael Brawne [1] likened this cyclic optimization process (as applied to architectural design) to Karl Popper's theory of the scientific method. The optimization process can be automated using computers using techniques including genetic algorithms [2] and simulated annealing [3].

Form finding techniques rely on physical or numerical models to automatically generate the form. The model must have different, but analogous, properties to the structure being designed and classic examples include Antoni Gaudí's hanging models for the masonry vaults of the Church of Colònia Güell [4] and Frei Otto's soap film models for fabric, cable net and gridshell structures. Form finding techniques do not produce an 'optimum form', but a 'good form'. However in practice the difference between optimization and form finding is arbitrary—one would expect form finding to be taken through a number of 'optimization' cycles involving analysis of the structure in its proposed final form in masonry or fabric.

Masonry shells can only work in compression and a number of numerical techniques have been developed for finding their geometry to achieve a specified stress state under dead load [5–12]. In this paper we propose the use of a variable isotropic stress state where the membrane stress is uniform in all directions with no shear stress, but the value of the stress varies from point to point.

There is no particular reason why the compressive membrane stress should be isotropic, but it could be argued that an isotropic stress is in some ways optimum. This mirrors the argument that a minimal surface is the best shape for a cable net or fabric structure. Certainly we want to avoid loss of compression in a masonry structure leading to cracking or loss of tension in a tension structure leading to wrinkling. Thus we want the radius of Mohr's circle for stress [13] to be less than the mean stress. The simplest case is to set the radius of Mohr's circle equal to zero corresponding to an isotropic stress. Note that Mohr's circle construction can be applied to any symmetric second order tensor, for example surface curvature [14].

Imposing the condition that the stress state should be isotropic also has the effect of avoiding undue stress concentrations. In general this is a good thing, but there are circumstances where one wants a concentration of stress or force, for example at a point support or at a boundary arch. However a boundary arch can be modelled as a separate entity leaving the state of stress in the rest of the shell isotropic.

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We will use the expression 'surface tension' to denote the value of the isotropic membrane stress expressed as a force per unit length. If the stress is compressive, then the surface tension is negative. It is often thought that the surface tension in a soap film is constant, but if this were the case it would not be possible for a soap film to carry its own weight. A vertical soap film must have a higher surface tension at the top than at the bottom. The situation is analogous to hydrostatic pressure that must increase with depth.

Even without gravity surface tension must vary with film thickness. Imagine a soap film with slight fluctuations in thickness. The surface tension must be greater where the film is thinner to pull fluid from thicker areas to ensure stability [15].

In the following sections we present the theoretical analysis of an isotropic membrane stress under gravity loads. We then give an example of the solution of the equations using the finite element method. Finally we formulate and illustrate the use of a particle method for numerical simulations.

2. Theoretical analysis

2.1. Geometric preliminaries

The methods described in Sections 2.1 and 2.2 are based on those in Green and Zerna [16], but with some changes in notation.

Consider a surface described by the position vector

$$\mathbf{r}\left(\theta^1,\theta^2\right)=x\left(\theta^1,\theta^2\right)\mathbf{i}+y\left(\theta^1,\theta^2\right)\mathbf{j}+z\left(\theta^1,\theta^2\right)\mathbf{k}. \tag{1}$$

$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the Cartesian axes and θ^1 and θ^2 are the surface parameters or coordinates replacing the u and v which are often used. Note that the 1 and 2 are not exponents.

The covariant base vectors are

$$\mathbf{g}_i=\frac{\partial\mathbf{r}}{\partial\theta^i}=\frac{\partial x}{\partial\theta^i}\mathbf{i}+\frac{\partial y}{\partial\theta^i}\mathbf{j}+\frac{\partial z}{\partial\theta^i}\mathbf{k} \tag{2}$$

in which i is equal to 1 or 2. $\mathbf{g}_1$ and $\mathbf{g}_2$ are tangent to the surface in the directions of increasing θ^1 and θ^2 respectively. Note that they are in general not unit vectors, nor are they perpendicular to each other.

The components of the metric tensor are

$$g_{ij}=\mathbf{g}_i\cdot\mathbf{g}_j \tag{3}$$

and the square of the distance between adjacent points on the surface is

$$\begin{aligned}\delta s^2&=\left(\sum_{i=1}^{2}\frac{\partial\mathbf{r}}{\partial\theta^i}\delta\theta^i\right)\cdot\left(\sum_{j=1}^{2}\frac{\partial\mathbf{r}}{\partial\theta^j}\delta\theta^j\right)\\&=\sum_{j=1}^{2}\sum_{i=1}^{2}g_{ij}\delta\theta^i\delta\theta^j=g_{ij}\delta\theta^i\delta\theta^j.\end{aligned} \tag{4}$$

The summations in the right hand side of this expression are implied by the Einstein summation convention. This expression for δs^2 is known as the first fundamental form and therefore g_{ij} are also known as the coefficients of the first fundamental form.

The quantity

$$g=g_{11}g_{22}-g_{12}^2 \tag{5}$$

and the unit normal,

$$\mathbf{n}=\frac{\mathbf{g}_1\times\mathbf{g}_2}{|\mathbf{g}_1\times\mathbf{g}_2|}=\frac{\mathbf{g}_1\times\mathbf{g}_2}{\sqrt{g}}. \tag{6}$$

Note that g is not a scalar in that it is a property of the coordinate system, rather than something with physical meaning.

The contravariant base vectors $\mathbf{g}^j$ also lie in the plane of the surface. They are defined by

$$\begin{aligned}&\mathbf{g}_i\cdot\mathbf{g}^j=\delta_i^j\\&\mathbf{n}\cdot\mathbf{g}^j=0\end{aligned} \tag{7}$$

in which the Kronecker deltas, $\delta_i^j=0$ if $i\neq j$ and $\delta_i^j=1$ if $i=j$. Thus $\mathbf{g}^1$ is perpendicular to both $\mathbf{g}_2$ and $\mathbf{n}$ and its magnitude is such that $\mathbf{g}_1\cdot\mathbf{g}^1=1$.

The contravariant components of the metric tensor are

$$g^{ij}=\mathbf{g}^i\cdot\mathbf{g}^j \tag{8}$$

and a vector can be expressed as

$$\mathbf{v}=v^i\mathbf{g}_i+v\mathbf{n}=v_i\mathbf{g}^i+v\mathbf{n} \tag{9}$$

in which

$$\begin{aligned}&v^i=g^{ij}v_j\\&v_i=g_{ij}v^j.\end{aligned} \tag{10}$$

Again note the use of the summation convention in (9) and (10).

Finally, the coefficients of the second fundamental form are

$$b_{ij}=b_{ji}=\frac{\partial\mathbf{g}_i}{\partial\theta^j}\cdot\mathbf{n}=\frac{\partial\mathbf{g}_j}{\partial\theta^i}\cdot\mathbf{n}=-\mathbf{g}_j\cdot\frac{\partial\mathbf{n}}{\partial\theta^i} \tag{11}$$

and the second fundamental form itself is

$$\delta\mathbf{r}\cdot\delta\mathbf{n}=-b_{ij}\delta\theta^i\delta\theta^j. \tag{12}$$

b_{ij} tell us about how the direction of the normal changes as we move about on the surface, in other words, about the curvature of the surface.

b_{ij} and g_{ij} are not independent, they are linked by the Gauss–Codazzi–Mainardi equations which ensure that the surface fits together.

2.2. The membrane equilibrium equations for shell and tension structures

We are now in a position to define the membrane stress tensor $\boldsymbol{\sigma}=\sigma^{ij}\mathbf{g}_i\mathbf{g}_j$ by

$$\delta\mathbf{f}=\epsilon_{ik}\sigma^{ij}\mathbf{g}_j\delta\theta^k \tag{13}$$

in which $\delta\mathbf{f}$ is the element of force crossing the imaginary cut $\delta\mathbf{r}=\mathbf{g}_k\delta\theta^k$. $\epsilon_{12}=-\epsilon_{21}=\sqrt{g}$ and $\epsilon_{11}=0$ and $\epsilon_{22}=0$ are the components of the Levi-Civita permutation pseudotensor. Note that we are not yet making the assumption that the membrane stress is isotropic.

Eq. (13) makes a bit more sense when written out in full:

$$\delta\mathbf{f}=\sqrt{g}\left(\sigma^{11}\delta\theta^2-\sigma^{21}\delta\theta^1\right)\mathbf{g}_1+\sqrt{g}\left(\sigma^{12}\delta\theta^2-\sigma^{22}\delta\theta^1\right)\mathbf{g}_2, \tag{14}$$

especially when compared to the equivalent relationship for plane stress in two dimensions in Cartesian coordinates:

$$\delta\mathbf{f}=\left(\sigma_x\delta y-\tau_{yx}\delta x\right)\mathbf{i}+\left(\tau_{xy}\delta y-\sigma_y\delta x\right)\mathbf{j}. \tag{15}$$

Equilibrium of moments about the surface normal tell us that the stress tensor is symmetric, $\sigma^{12}=\sigma^{21}$.

Adding the forces on a small quadrilateral of shell we have

$$\begin{aligned}&\frac{\partial}{\partial\theta^2}\left(\epsilon_{i1}\sigma^{ij}\mathbf{g}_j\left(-\delta\theta^1\right)\right)\delta\theta^2\\&\quad+\frac{\partial}{\partial\theta^1}\left(\epsilon_{i2}\sigma^{ij}\mathbf{g}_j\delta\theta^2\right)\delta\theta^1+\mathbf{w}\sqrt{g}\delta\theta^1\delta\theta^2=0\end{aligned} \tag{16}$$

where $\mathbf{w}$ is the load per unit surface area. Thus

$$\frac{\partial}{\partial\theta^i}\left(\sqrt{g}\sigma^{ij}\mathbf{g}_j\right)+\mathbf{w}\sqrt{g}=0. \tag{17}$$